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A Nonconvex Set Which Has the Unique Nearest Point Property

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There is a well-known problem in approximation theory as to whether or not every set in a Hilbert space that has the property that each point in the space has a unique nearest point in the set, is convex. This problem was first mentioned in Klee [1].

We shall construct a subset S of the real inner product space E of all real sequences having at most a finite number of nonzero terms, with inner product $(x, y) = \sum_i x_i y_i$, where $x = (x_1, x_2, ...)$, $y = (y_1, y_2, ...)$, and induced norm $\|x\| = \sqrt{(x, x)}$, such that

(1) S is closed and nonconvex;

(2) each point in E has a unique nearest point in S;

(3) S is not a sun; and

(4) the metric projection is continuous.

It is well known that if X is a finite dimensional Euclidean space and S is a subset of X such that each point in X has a unique nearest point in S, then S is a closed and convex set (Bunt [2]). Moreover, if X is a real Hilbert space and S is a boundedly compact subset of X having the unique nearest point property mentioned above, then S is convex (Vlaslov [3]). It should be noted that if S is a closed and convex set in a real Hilbert space, then S is a sun, the metric projection is continuous, and S is approximately compact. Indeed, each two of the above three statements are equivalent in a Hilbert space (Vlaslov [3]). There is a great deal of literature directly related to this problem and that so few are listed here is not intended to suggest that the other works are of any lesser significance or relevance.

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For each positive integer n let

$$E_n^- = \{x : x \in E, x_n \leqslant 0 \text{ and } x_i = 0 \text{ if } i > n\},$$
$$E_n^+ = \{x : x \in E, x_n > 0 \text{ and } x_i = 0 \text{ if } i > n\}$$

and $E_n = E_n^- \cup E_n^+$.

We can clearly identify E_n with Euclidean n dimensional space and shall, on occasion, write a point x in E_n as $(x_1, x_2, ..., x_n)$.

We shall proceed in three parts: the first part shall indicate the construction and how it came about, the second will establish the statements made in part one, and the third part will provide the computations needed in part two.

1. The Construction of the Set S

Let $\{\phi_1, \phi_2, ...\}$ be the standard orthonormal basis for E, i.e., for each positive integer n, ϕ_n is that sequence in E for which each term is zero except the nth term, which is one.

Step 0. Let $S_0 = \{-\phi_1\}$ and notice that each point in E_1^- has a unique nearest point in the closed set S_0 (see Fig. 1).

Step 1. We now have the task of constructing a closed nonconvex set S_1, such that $S_0 \subset S_1$, and also that each point in E_1 has a unique nearest point in S_1. We know that S_1 cannot be a subset of E_1, and hence shall construct S_1 as a subset of E_2 (see Fig. 2).

Let us now select $2\phi_1$ as a point that we shall include in S_1 so that each point in E_1, greater than or equal to one, has $2\phi_1$ as the unique nearest point, leaving us with only the segment $(0, 1)$ to contend with.

The problem now is to determine a subset S_1 of E_2 such that each point in E_1 has a unique nearest point in S_1 and such that $S_0 \subset S_1$. To this end we start with the point $\frac{1}{2} = Q$ and select a point P in the upper plane that is directly above Q and closer to Q than either $-\phi_1$ or $2\phi_1$ is to Q. We now have a point labeled R that is equidistant from both P and $-\phi_1$. This now requires that we find a point that is in the upper plane that is closer to R than either $-\phi_1$ or P. This rather unorganized process now suggests what is to be done (see Fig. 3).

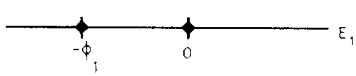

Figure 1

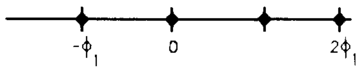

FIGURE 2

That is, we need to determine a function f_1 on the interval $[-1, 2]$ whose graph is the desired set. We do this in the following manner (see Fig. 4).

What we shall do is determine a function f_1, defined on the number interval $[-1, 2]$, such that each point in $[0, 1]$ has a unique nearest point in f_1. To do this, suppose that there is such a function f_1. Let $P=(x, f_1(x))$ be a point of f_1 and consider the line normal to f_1 at P, which then has slope $-1/f_1'(x)$ (assuming that f_1 is differentiable). The equation of the line normal to f_1 at P is

$$y(t)=(-1/f_1'(x))[t-x]+f_1(x) \qquad \text{for} \quad -\infty \leqslant t \leqslant \infty$$

and this line intersects E_1 at the point $Q=(t_0, 0)$, i.e., when

$$f_1(x)\, f_1'(x)=t_0-x.$$

Define g_1 by $g_1(z)=\frac{1}{3}(z+1)$ for $-1 \leqslant z \leqslant 2$ and note that g_1 is an increasing homeomorphism from $[-1, 2]$ onto $[0, 1]$.

Let $t_0=g_1(x)=\frac{1}{3}(x+1)$ and hence

$$f_1(x)\, f_1'(x)=\tfrac{1}{3}(x+1)-x=\tfrac{1}{3}(1-2x),$$

from which it follows that

$$(f_1^2(x))'=2(1-2x)/3,$$

or

$$f_1^2(x)=\tfrac{2}{3}[x-x^2]+\text{constant}.$$

We require that $f_1(-1)=0$, and thus

$$f_1^2(x)=\tfrac{2}{3}[2+x-x^2] \qquad \text{for} \quad -1 \leqslant x \leqslant 2$$

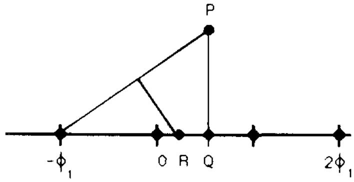

FIGURE 3

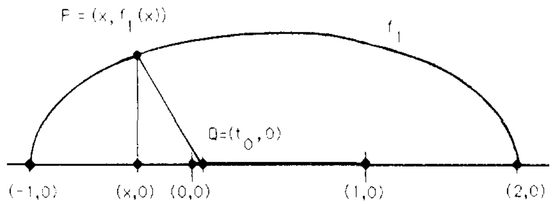

FIGURE 4

and

$$g_1(x) = \tfrac{1}{3}(x+1) \qquad \text{for} \quad -1 \leqslant x \leqslant 2.$$

One could notice that f_1 is an ellipse. That the graph of f_1 has the desired properties will be demonstrated later. The choice of g_1 was made on the basis of simplicity, but, as we shall see, it plays an important role in all that follows.

Let us assume that f_1 has the desired properties, i.e., that each point in E_1 has a unique nearest point in the set S_1 which is the graph of f_1. By the way f_1 was constructed, it follows that each point in the region bounded by f_1 and E_1 has a unique nearest point in S_1. Since f_1 is an ellipse it follows then that each point in E_2^+ has a unique nearest point in S_1. Form the mirror image of the set S_1 with respect to E_1, and designate this set as

$$S_1 = \{(x, -f_1(x)\text{: } -2 \leqslant x \leqslant 1\}.$$

The problem now is to find a set S_2 in E_3 such that each point in E_2 has a unique nearest point in S_2, and such that if a point Q in E_2 has unique nearest point P in S_1, then P is in S_2 and is the unique nearest point in S_2 to Q. To visualize the problem consider Figs. 5 and 6 and, in particular, their grey regions.

What has been constructed so far is only the set S_1 and what is needed is the entire closed curve in E_2, the grey region Y_2, and the surface S_2. These

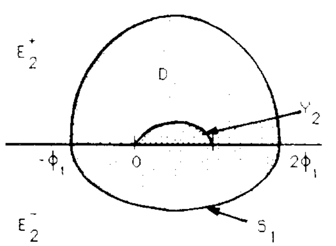

FIGURE 5

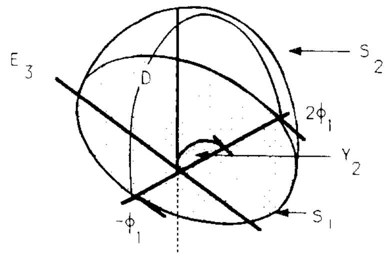

FIGURE 6

three are closely intertwined in the following sense. We need the surface S_2 to have the property that

(1) Each normal line intersects the plane only in the region Y_2,

(2) no two normal lines intersect in the region contained by S_2 and the region D in the plane, and

(3) there is a homeomorphism G_2 of the region D in the plane onto the grey region Y_2, such that if P is a point in the surface S_2 and x is the point in the subset D in the plane directly below P, then G_2 maps x onto the point of intersection of the grey region Y_2 and the line normal to S_2 at P.

What we shall show is that the point P in S_2 is the unique nearest point in S_2 to the point $G_2(x)$ in Y_2, and also to each point in the line interval $[G_2(x), P]$.

To set matters on a firmer footing, what we need to determine is a function F defined on the closure of a convex region D in the plane, where S_1 forms part of the boundary of D, and from this function, a pair of functions $g_{1,2}$ and $g_{2,2}$ each from the region D to the numbers which are determined by the function F in the following manner:

$$g_{1,2}(x, y) = x + \frac{1}{2}\frac{\partial F^2}{\partial x}(x, y)$$

$$g_{2,2}(x, y) = y + \frac{1}{2}\frac{\partial F^2}{\partial y}(x, y).$$

Let $G_2 = (g_{1,2}, g_{2,2})$, which then is a function from the region D into the plane.

A point P in S_2 has coordinates $(x, y, -F(x, y))$ and the point of intersection of the line normal to S_2 at P and the plane, is the point $G_2(x, y) = (g_{1,2}(x, y), g_{2,2}(x, y))$.

The conditions that we need to impose on F are that

(1) F^2 be differentiable (except at those points of F that are in the plane),

(2) $F(x, -f_1(x)) = 0$ for $-1 \leqslant x \leqslant 2$,

(3) the function G_2 be a homeomorphism of the entire region D in the plane onto the grey region Y_2, and

(4) $F(x, a(x) f_1(x)) = 0$ for $-1 \leqslant x \leqslant 2$ where a is the function on $[-1, 2]$ that, when multiplied by f_1 determines the upper boundary of the region in the plane.

Here is the last figure to assist in following the example, and in particular to see that the set S is not a sum (Fig. 7). Perhaps it is worth noting that some of the ideas that led to this example are to be found in Johnson [4]. A summary of much that has been done in the finite dimensional case can be found in Kelly [5].

The above should be understood in order to follow in a geometric way, the many computations that are to follow, as well as to gain a feeling for what the set S that we shall construct "looks" like. If an understanding is had at this point, then one should sense that the final set S, which contains the set S_2, is not a sun. It is interesting to note that if a Hilbert space contains a nonconvex Chebyshev set, then it contains one whose complement is bounded and convex (Asplund [6]).

We now begin. What occurs first are the technical statements that are to be established and then, using these statements, the proofs of the assertions made in the beginning paragraphs.

Let us define the following:

$$a_0 = 2, \qquad A_0 = 1,$$

$$F_0 = 1, \qquad L_0 = 1,$$

$$d_1 = \{x_1 \colon -F_0 \leqslant x_1 \leqslant a_0 F_0\}$$

$$D_1 = \{x_1 \phi_1 \colon x_1 \in d_1\},$$

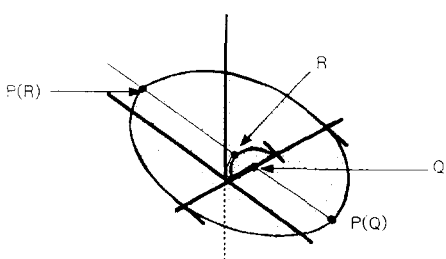

FIGURE 7

$$h_1(y) = x_1 \qquad \text{for} \quad y = x_1\phi_1 \in D_1,$$

$$L_1(x_1) = a_0 F_0^2 + (a_0 - 1)\, F_0 x_1 - x_1^2 \colon x_1 \in d_1,$$

$$F_1^2(x_1) = 2L_1(x_1)/[a_0 + 1] \colon x_1 \in d_1,$$

$$S_1 = \{x_1\phi_1 - F_1(x_1)\,\phi_2 \colon x_1 \in d_1\}$$

$$g_{1,1}(x_1) = x_1 + [(a_0 - 1)\, F_0 - 2x_1]/[a_0 + 1] \colon x_1 \in d_1,$$

$$G_1(y) = g_{1,1}(h_1(y))\,\phi_1 \colon y \in D_1,$$

Y_1 = image of D_1 under G_1,

I_1 = bounded region determined by S_1 and D_1.

STATEMENT 1 (see Fig. 8). 1.1. *D_1 is a bounded, closed and convex set.*

1.2. $D_1 \subseteq E_1$.

1.3. *G_1 is a homeomorphism.*

1.4. *I_1 is convex.*

1.5. *Each point Q in Y_1 has a unique nearest point P in S_1, and each point in S_1 is the unique nearest point in S_1 for some point in Y_1.*

1.6. *Each point in E_2^- has a unique nearest point in S_1.*

1.7. $S_0 \subseteq S_1$.

1.8. *If W is in E_1^- and P is the unique nearest point in S_1 to W, then P is in S_0 to W.*

In what follows we shall use the following notation:

$$D_i F(x_1, ..., x_n) = \frac{\partial F}{\partial x_i}(x_1, ..., x_n) \qquad 1 \leqslant i \leqslant n$$

and

$$D_{j,i} F(x_1, ..., x_n) = \frac{\partial^2 F}{\partial x_j\, \partial x_i}(x_1, ..., x_n) \qquad 1 \leqslant i, j \leqslant n.$$

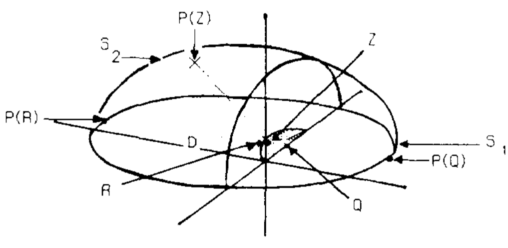

FIGURE 8

Step 2. We now have the task of constructing a closed and nonconvex set S_2 such that $S_1 \subseteq S_2$, and each point in E_2 has a unique nearest point in S_2.

Let A_1 be a positive number to be chosen later and

$$
\begin{aligned}
&a_1(x) = 1 + A_1 L_1(x_1): x_1 \in d_1,\\
&d_2 = \{(x_1, x_2): x_1 \in d_1,\ -F_1(x_1) \leqslant x_2 \leqslant a_1 F_1(x_1)\},\\
&D_2 = \{x_1\phi_1 + x_2\phi_2: (x_1, x_2) \in d_2\},\\
&h_{2,1}(y) = x_1: \text{ for } y = x_1\phi_1 + x_2\phi_2 \in D_2,\\
&h_{2,2}(y) = x_2: \text{ for } y = x_1\phi_1 + x_2\phi_2 \in D_2,\\
&h_2(y) = (h_{2,1}(y), h_{2,2}(y)): y \in D_2,\\
&L_2(x_1, x_2) = a_1 F_1^2(x_1) + (a_1 - 1) F_1(x_1) x_2 - x_2^2: (x_1, x_2) \in d_2,\\
&F_2^2(x_1, x_2) = 2L_2(x_1, x_2)/[a_1(x_1) + 1]: (x_1, x_2) \in d_2,\\
&S_2 = \{x_1\phi_1 + x_2\phi_2 - F_2(x_1, x_2)\,\phi_3: (x_1, x_2) \in d_2\}\\
&g_{2,1}(x_1, x_2) = x_1 + [F_1(x_1) + x_2)/(a_1(x_1) + 1)]^2 D_1 a_1(x_1)\\
&\qquad\qquad + [2a_1(x_1) + (a_1(x_1) - 1)\, x_2/F_1(x_1)]\\
&\qquad\qquad \times [g_{1,1}(x_1) - x_1]/[a_1(x_1) + 1]: (x_1, x_2) \in d_2,\\
&g_{2,2}(x_1, x_2) = x_2 + [(a_1 - 1) F_1(x_1) - 2x_2]/[a_1(x_1) + 1]: (x_1, x_2) \in d_2,\\
&G_2(y) = g_{2,1}(h_2(y))\,\phi_1 + g_{2,2}(h_2(y))\,\phi_2: \text{for each } y = x_1\phi_1 + x_2\phi_2 \in D_2,\\
&Y_2 = \text{image of } D_2 \text{ under } G_2,\\
&I_2 = \text{bounded region determined by } S_2 \text{ and } D_2,\\
&JG_2 = |D_i g_{2,j}|, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
T_2(\hat{h}_1, \hat{h}_2, x_1, x_2) = \sum_{i=1}^{2} &\hat{h}_i D_{i,i} F_2(x_1, x_2)\\
&+ 2 \sum_{i<j} \hat{h}_i \hat{h}_j D_{i,j} F_2(x_1, x_2): (x_1, x_2) \in d_2.
\end{aligned}
$$

Notice that

$$g_{2,1}(x_1, x_2) = x_1 + D_1 F_2^2(x_1, x_2)/2$$

and

$$g_{2,2}(x_1, x_2) = x_2 + D_2 F_2^2(x_1, x_2)/2.$$

STATEMENT 2. *There is a positive number* A_1^* *such that if* $A_1^* > A_1 > 0$, *then*

2.1. *D_2 is a bounded, closed, and convex set*;

2.2. $D_2 \subseteq E_2$;

2.3. *G_2 is a homeomorphism*;

2.4. *I_2 is convex*;

2.5. *each point Q in Y_2 has a unique nearest point P in S_2, and each point in S_2 is the unique nearest point in S_2 for some point in Y_2*;

2.6. *each point in E_3^- has a unique nearest point in S_2*;

2.7. $S_1 \subseteq S_2$;

2.8. *if W is in E_2^- and P is the unique nearest point in S_2 to W, then P is in S_1 and is the unique nearest point in S_1 to W*;

2.9. $F_2^3(x_1, x_2)\, T_2(\hat{h}_1, \hat{h}_2, x_1, x_2)$
$$\leqslant -[\textstyle\sum_{i=1}^2 \hat{h}_i[g_{2,i}(x_1, x_2) - x_i]]^2 - [F_2^2(x_1, x_2)\, 3^{-1}] \textstyle\sum_{i=1}^2 \hat{h}_i^2 \colon (x_1, x_2) \in d_2;$$ *and*

2.10. $JG_2 = [\prod_{i=1}^1 L_i]\, J_2$, *where J_2 is positive on D_2*.

Let us now suppose that we have proceeded for n steps.

Step $n+1$. Let A_n be a positive number and

$$a_n(x_1, ..., x_n) = 1 + A_n L_n(x_1, ..., x_n) \colon (x_1, ..., x_n) \in d_n,$$

$$d_{n+1} = \{(x_1, ..., x_{n+1}) \colon (x_1, ..., x_n) \in d_n,\ -F_n(x_1, ..., x_n) \leqslant x_{n+1} \leqslant a_n F_n(x_1, ..., x_n)\},$$

$$D_{n+1} = \left\{\sum_{i=1}^{n+1} x_i \phi_i \colon (x_1, ..., x_{n+1}) \in d_{n+1}\right\}.$$

For $1 \leqslant i \leqslant n+1$,

$$h_{n+1,i}(y) = x_i \colon y \in D_{n+1},$$

$$h_{n+1}(y) = (h_{n+1,1}(y), ..., h_{n+1,n+1}(y)) \colon y \in D_{n+1},$$

$$L_{n+1}(x_1, ..., x_{n+1}) = a_n F_n^2(x_1, ..., x_n) + (a_n - 1)\, F_n(x_1, ..., x_n)\, x_{n+1} - x_{n+1}^2 \colon (x_1, ..., x_{n+1}) \in d_{n+1},$$

$$F_{n+1}^2(x_1, ..., x_{n+1}) = 2L_{n+1}(x_1, ..., x_{n+1})/[a_n(x_1, ..., x_n) + 1] \colon (x_1, ..., x_{n+1}) \in d_{n+1},$$

$$S_{n+1} = \left\{\sum_{i=1}^{n+1} x_i \phi_i - F_{n+1}(x_1, ..., x_{n+1})\, \phi_{n+2} \colon (x_1, ..., x_{n+1}) \in d_{n+1}\right\},$$

for $1 \leqslant i \leqslant n$,

$$
\begin{aligned}
g_{n+1,i}&(x_1, ..., x_{n+1}) \\
&= x_i + [(F_n(x_1, ..., x_n) + x_{n+1})/(a_n(x_1, ..., x_n) + 1)]^2 D_i a_n(x_1, ..., x_n) \\
&\quad + [2a_n(x_1, ..., x_n) + (a_n(x_1, ..., x_n) - 1)\, x_{n+1}/F_n(x_1, ..., x_n)] \\
&\quad \times [g_{n,i}(x_1, ..., x_n) - x_i]/[a_n(x_1, ..., x_n) + 1]\colon (x_1, ..., x_{n+1}) \in d_{n+1},
\end{aligned}
$$

$$
\begin{aligned}
g_{n+1,n+1}&(x_1, ..., x_{n+1}) \\
&= x_{n+1} + [(a_n - 1)\, F_n(x_1, ..., x_n) - 2x_{n+1}]/[a_n(x_1, ..., x_n) + 1]\colon \\
&\quad (x_1, ..., x_{n+1}) \in d_{n+1},
\end{aligned}
$$

$$G_{n+1}(y) = \sum_{i=1}^{n+1} g_{n+1,i}(h_{n+1}(y))\, \phi_i\colon y \in D_{n+1},$$

Y_{n+1} = image of D_{n+1} under G_{n+1},

I_{n+1} = bounded region determined by S_{n+1} and D_{n+1},

$JG_{n+1} = |D_i g_{n+1,j}|$,

$$
\begin{aligned}
T_{n+1}&(\hat{h}_1, ..., \hat{h}_{n+1}, x_1, ..., x_{n+1}) \\
&= \sum_{i=1}^{n+1} \hat{h}_i^2 D_{i,i} F_{n+1}(x_1, ..., x_{n+1}) \\
&\quad + 2 \sum_{i=1}^{n+1} \hat{h}_i \hat{h}_j D_{i,j} F_{n+1}(x_1, ..., x_{n+1})\colon (x_1, ..., x_{n+1}) \in d_{n+1}.
\end{aligned}
$$

STATEMENT $n+1$. *There is a positive number $A_n^* > 0$ such that if $A_n^* > A_n > 0$, then*

$n+1.1$. *D_{n+1} is bounded, closed, and convex set;*

$n+1.2$. *$D_{n+1} \subseteq E_{n+1}$;*

$n+1.3$. *G_{n+1} is a homeomorphism;*

$n+1.4$. *each point Q in Y_{n+1} has a unique nearest point P in S_{n+1}, and each point in S_{n+1} is the unique nearest point in S_{n+1} for some point in Y_{n+1};*

$n+1.5$. *I_{n+1} is convex;*

$n+1.6$. *each point in E_{n+2} has a unique nearest point in S_{n+1};*

$n+1.7$. *$S_n \subseteq S_{n+1}$;*

$n+1.8$. *if W is in E_n^- and P is the unique nearest point in S_{n+1} to W, then P is in S_n and is the unique nearest point in S_n to W;*

$n+1.9.$ $$F^3_{n+1}(x_1, ..., x_{n+1})\, T_{n+1}(\hat{h}_1, ..., \hat{h}_{n+1}, x_1, ..., x_{n+1}) \leqslant -\left(\sum_{i=1}^{n+1} \hat{h}_i[g_{n+1,i}(x_1, ..., x_{n+1}) - x_i]\right)^2 - [F^2_{n+1}(x_1, ..., x_{n+1})\, 3^{-n}] \sum_{i=1}^{n+1} \hat{h}_i^2:$$

$(x_1, ..., x_{n+1}) \in d_{n+1}$; *and*

$n+1.10.$ $JG_{n+1} = (\prod_{i=1}^{n} L_i)\, J_{n+1}$, *where* J_{n+1} *is positive on* D_{n+1}.

Let us sum up what we have thus far. If $A_1, A_2, ...$ is a positive number sequence such that statement n is true for every positive integer n, then $S = \bigcup_{i=0}^{\infty} S_i$ is a nonconvex subset of E. Moreover, if W is a point in E, then there is a positive integer n such that W is in E_n and a unique nearest point P in S_n to W. This point P is then the unique nearest point in S to W; moreover, if W is not in S, then W cannot be a limit point of S and hence S is closed.

To show that the metric projection P is continuous, consider a point y in Y and a point sequence $\{y_i\}_i$ in Y which converges to y. Associated with each y_i is the point $P(y_i)$ in S, and associated with y is the point $P(y)$. Notice that if z is in Y and $P(z)$ is in E_n, then $G_{n+k}(P(z)) = z$ for each positive integer k.

For $x = (x_1, x_2, ..., x_n)$ and k a positive integer define

$$[x]^k = \begin{cases} (x_1, x_2, ..., x_k) & \text{if } k \leqslant n \\ (x_1, x_2, ..., x_n, ..., x_k) & \text{if } k \geqslant n \text{ where each of } x_{n+1}, ..., x_k \text{ is } 0. \end{cases}$$

Notice that if $y = (y_{1,0}, ..., y_{n,0})$ and k is a nonnegative integer, then $\{[y_i]^{n+k}\}_i$ converges to y, and since G_{n+k+1} is a homeomorphism the point sequence $\{P[y_i]^{n+k}\}_i$ converges to $P(y)$. Moreover, for each i, the point sequence $\{P[y_i]^{n+k}\}_n$ converges to $P(y_i)$. Thus we have that $\{P(y_i)\}_i$ converges coordinatewise to $P(y)$.

It is clear that

$$\lim_{i \to \infty} \|y_i - P(y_i)\| = \|y - P(y)\|,$$

and since $\lim_{i \to \infty} y_i = y$ it follows that $\lim_{i \to \infty} \|y - P(y_i)\| = \|y - P(y)\|$.

Recall that $\{P(y_i)\}_i$ converges coordinatewise to $P(y)$ which, when coupled with the above, implies that $\{P(y_i)\}_i$ converges to $P(y)$. Thus P is continuous on Y.

If x is a point in $D = \bigcup D_i$, then x is in a unique interval $[y, P(y)]$, where y is in Y and $P(y)$ is the unique nearest point in S to y. Recall that $P(x) = P(y)$ and, since P is continuous at y, it follows that P is continuous at x.

To show that S is not a sun consider Fig. 7 and in particular the point indicated by Q. Notice that each point that is past Q in the order from $P(Q)$ to Q, in the half ray starting from $P(Q)$ and containing the point Q, does not have $P(Q)$ as a unique nearest point in S. Hence S is not a sun.

If one recalls the result of Asplund about complements, and considers the set $S' = \bigcup I_i$ and the closure S'' of S', then one can see a closed convex set whose boundary is S, thus the complement of S' is a closed nonconvex set having the property that each point in E has a unique nearest point in it and whose complement is convex. By a careful selection of the $\{A_i\}_i$ sequence, it is possible to show that S' is bounded. If we are allowed to digress further, then we may also notice the possibility of having many pairwise disjoint copies of S' dispersed throughout the space, and then forming the complement to have a rather spongy set with the property that each point in the space has a unique nearest point in the sponge.

2. Proofs of Statements

Statement 1. Substatements 1.1 and 1.2 are obviously true.

Substatement 1.3. For $-1 \leqslant x_1 \leqslant 2$, $g_{1,1}(x_1) = (1 + x_1)/3$ and hence $g_{1,1}$ is a homeomorphism of $[-1, 2]$ onto $[0, 1]$ from which it follows that G_1 is a homeomorphism of D_1 onto Y_1.

Substatement 1.4. It is sufficient to observe that $D_{1,1}F_1(x_1) < 0$ for $-1 < x < 2$ and thus F_1 is concave from which it follows that I_1 is convex.

Substatement 1.5. For each point Q in Y_1 there is a unique point M in D_1 such that $G_1(M) = Q = z_1\phi_1$. Let $h_1(M) = x_1$, $P = x_1\phi_1 - F_1(x_1)\,\phi_2$, and $P' = x_1'\phi_1 - F_1(x_1')\,\phi_2$, where x_1' is a number in d_1 distinct from x_1.

We shall show that

$$\|P - Q\| < \|P' - Q\|.$$

A straightforward calculation shows that

$$\|P' - Q\|^2 - \|P - Q\|^2 = (x_1' - z_1)^2 + F_1^2(x_1') - (x_1 - z_1)^2 - F_1^2(x_1).$$

There is a number c in d_1 such that

$$F_1^2(x_1') = F_1^2(x_1) + D_1 F_1^2(x_1)(x_1' - x_1) + D_{1,1} F_1^2(c)(x_1' - x_1)^2/2.$$

Combining this with $D_1 F_1^2(x_1) = 2(1 - 2x_1)/3$ and $D_{1,1} F_1^2(c) = -\frac{4}{3}$ we have that

$$\begin{aligned}\|P' - Q\|^2 - \|P - Q\|^2 = (x_1' - z_1)^2 + 2(1 - 2x_1)(x_1' - x_1)/3 \\ - 2(x_1' - x_1)^2/3 - (x_1 - z_1)^2.\end{aligned}$$

Recall that $z_1 = g_{1,1}(x_1) = (1 + x_1)/3$ and observe that

$$(x_1' - z_1)^2 = (x_1' - x_1)^2 + 2(x_1' - x_1)(x_1 - z_1) + (x_1 - z_1)^2,$$

from which it follows that

$$\|P' - Q\|^2 - \|P - Q\|^2 = (x_1' - x_1)^2/3$$

and hence $\|P' - Q\|^2 > \|P - Q\|^2$.

Substatement 1.6. If W is in E_2^- and W is not in I_1, then W has a unique nearest point in S_1. If W is in I_1, then there is a point Q in Y_1 and a point P in S_1 such that P is the unique nearest point in S_1 to Q and W is in the interval $[Q, P]$. Let C be the ball centered at W with radius $\|P - W\|$. If C contains a point of S_1 distinct from P, then such a point would be closer to Q than P is to Q which is a contradiction. Hence P is the unique nearest point in S_1 to W.

Substatements 1.7 and 1.8 are obviously true.

Statement 2. Substatement 2.1. Clearly D_2 is closed and bounded and $d_2 = \{(x_1, x_2)\colon\ x_1 \in d_1,\ \ -F_1(x_1) \leqslant x_2 \leqslant 0\}$ is closed and convex since I_1 is convex; let $d_2^+ = \{(x_1, x_2)\colon\ x_1 \in d_1,\ \ 0 \leqslant x_2 \leqslant a_1 F_1(x_1)\}$. A straightforward calculation shows that

$$D_{1,1} a_1 F_1(x_1) = [-1/F_1^3][1 + A_1 F_1^3(4L_1 - 9/2)](x_1) \quad \text{for} \quad -F_0 < x_1 < a_0 F_0$$

and hence there is a number $A_1' > 0$ such that if $A_1' > A_1 > 0$, then $D_{1,1} a_1 F_1(x_1) < 0$ for $-F_0 < x_1 < a_0 F_0$, from which it follows that d_2^+ is convex. Also $d_2 = d_2^+ \cup d_2^-$, $d_2^+ \cap d_2^- = \{(x_1, 0)\colon x_1 \in d_1\}$, the projection of d_2^+ onto d_1 is the projection of d_2^- onto d_1 is d_1, and thus d_2 is closed and convex from which it follows that D_2 is a convex, closed, and bounded set.

Substatement 2.2 is clearly true.

Substatement 2.3. We shall employ Lemma A which is found in Section 3. Let us first show that G_2 is reversible, i.e., invertible, on the boundary of D_2.

If $x_2 = -F_1(x_1)$ for x_1 in d_1, then

$$g_{2,1}(x_1, -F_1(x_1)) = g_{1,1}(x_1)$$

and

$$g_{2,2}(x_1, -F_1(x_1)) = 0.$$

Therefore if $x_2 = -F_1(x_1)$, $G_2 = G_1$ and hence is reversible. If $x_2 = a_1 F_1(x_1)$ for x_1 in d_1, then

$$g_{2,1}(x_1, a_1 F_1(x_1)) = x_1 + (1 - 2x_1)(3a_1(x_1) - 2)/3$$

and

$$g_{2,2}(x_1, a_1 F_1(x_1)) = A_1 L_1(x_1) F_1(x_1).$$

Moreover,

$$g'_{2,1}(x_1, a_1 F_1(x_1) = 1/3 + A_1((1-2x_1)^2 - 2L_1(x_1))$$

and hence there is a number $A_1'' > 0$ such that if $A_1'' > A_1 > 0$ then

$$1/9 < g'_{2,1}(x_1, a_1 F_1(x_1)) < 1/2 \qquad \text{for} \quad x_1 \text{ in } d_1.$$

Notice also that $g_{2,2}(x_1, a_1 F_1(x_1)) \geqslant 0$ for any $A_1 > 0$ and finally that

$$g_{2,1}(-1, a_1 F_1(-1)) = 0$$

and

$$g_{2,1}(2, a_1 F_1(2)) = 1.$$

Thus we have that there is a number $A_1'' > 0$ such that if $A_1'' > A_1 > 0$, then G_2 restricted to the boundary of D_2 is invertible.

Let us now show that G_2 has a local inverse at each interior point of D_2 (considered as a subset of a two-dimensional space).

Since $D_2 g_{2,1} = D_1 g_{2,2}$ it follows that JG_2, the Jacobian of G_2, is $D_1 g_{2,1} D_2 g_{2,2} - (D_2 g_{2,1})^2$. In Section 3 we shall show that there is a number $A > 0$ such that if $A > A_1 > 0$, then $|JG_2| > 0$ on the interior of D_2 and hence G_2 has a local inverse at each interior point of D_2.

Thus the hypothesis of Lemma A is satisfied and hence G_2 is a homeomorphism of D_2 onto Y_2.

Substatement 2.4. To show that the region bounded by S_2 and D_2 is convex it is sufficient to show that F_2 is concave, i.e.,

$$2F_2(x_1, x_2) \geqslant F_2(x_1 + \hat{h}_1, x_2 + \hat{h}_2) + F_2(x_1 - \hat{h}_1, x_2 - \hat{h}_2).$$

Let $x = (x_1, x_2)$ and $\hat{h} = (\hat{h}_1, \hat{h}_2)$. There are points c and c' between x and $x + \hat{h}$ and x and $x - \hat{h}$, respectively, such that

$$F_2(x+\hat{h}) = F_2(x) + \sum_{i=1}^{2} \hat{h}_i D_i F_2(x) + \sum_{i=1}^{2} \hat{h}_i^2 D_{i,i} F_2(c)/2$$
$$+ \sum_{i<j}^{2} \hat{h}_i \hat{h}_j D_{i,j} F_2(c)$$

and

$$F_2(x-\hat{h}) = F_2(x) - \sum_{i=1}^{2} \hat{h}_i D_i F_2(x) + \sum_{i=1}^{2} \hat{h}_i^2 D_{i,i} F_2(c')/2$$
$$+ \sum_{i<j}^{2} \hat{h}_i \hat{h}_j D_{i,j} F_2(c').$$

Using the above we have that

$$\begin{aligned} 2[F_2(x+\hat{h}) + F_2(x-\hat{h}) - 2F_2(x)] = & \sum_{i=1}^{2} \hat{h}_i^2 D_{i,i} F_2(c) \\ & + 2 \sum_{i<j}^{2} \hat{h}_i \hat{h}_j^2 D_{i,j} F_2(c) \\ & + \sum_{i=1}^{2} \hat{h}_i^2 D_{i,i} F_2(c') \\ & + 2 \sum_{i<j}^{2} \hat{h}_i \hat{h}_j D_{i,j} F_2(c'). \end{aligned}$$

Let

$$T_2(\hat{h}, c) = \sum_{i=1}^{2} \hat{h}_i^2 D_{i,i} F_2(c) + 2 \sum_{i<j}^{2} \hat{h}_i \hat{h}_j D_{i,j} F_2(c)$$

for c in the interior of d_2 and $\hat{h}$ such that $c+\hat{h}$ is in the interior of d_2.

If we can show that $T_2(\hat{h}, c) \leqslant 0$, then the result is established. For $1 \leqslant i$, $j \leqslant 2$,

$$D_{i,j} F_2 = (D_{i,j} F_2^2)/2F_2 - (D_i F_2^2)(D_j F_2^2)/4F_2^3.$$

Using this we then have that

$$\begin{aligned} T_2(\hat{h}, c)\, F_2^3(c) = & -\left(\sum_{i=1}^{2} \hat{h}_i D_i F_2^2(c)/2 \right)^2 \\ & + F_2^2(c) \left(\sum_{i=1}^{2} \hat{h}_i^2 D_{i,i} F_2^2(c)/2 + 2 \sum_{i<j}^{2} \hat{h}_i \hat{h}_j D_{i,j} F_2^2(c)/2 \right). \end{aligned}$$

Let

$$t_2(\hat{h}, c) = \sum_{i=1}^{2} \hat{h}_i^2 D_{i,i} F_2^2(c)/2 + 2 \sum_{i<j}^{2} \hat{h}_i \hat{h}_j D_{i,j} F_2^2(c)/2,$$

and recall that for $1 \leqslant i, j \leqslant 2$,

$$D_i F_2^2(c)/2 = g_{2,i}(c) - c_i,$$
$$D_{i,i} F_2^2(c)/2 = D_i g_{2,i}(c) - 1$$

and if $j \neq i$,

$$D_{i,j} F_2^2(c)/2 = D_i g_{2,j}(c).$$

Thus

$$\begin{aligned} t_2(\hat{h}, c) &= \sum_{i=1}^{2} \hat{h}_i^2 (D_i g_{2,i}(c) - 1) + 2 \sum_{i<j}^{2} \hat{h}_i \hat{h}_j D_i g_{i,j}(c) \\ &= -\sum_{i=1}^{2} \hat{h}_i^2 (1 - D_i g_{2,i}(c)) \\ &\quad + \sum_{i=1}^{1} \sum_{j=i+1}^{2} 2\hat{h}_i \hat{h}_j D_i g_{2,j}(c) \end{aligned}$$

and

$$T_2(\hat{h}, c)\, F_2^3(c) = F_2^2(c)\, t_2(\hat{h}, c) - \left(\sum_{i=1}^{2} \hat{h}_i [g_{2,i}(c) - c_i] \right)^2.$$

Hence if we can show that $t_2(\hat{h}, c) \leqslant 0$ we will have that $T_2(\hat{h}, c) \leqslant 0$.

From Lemma I we have that $t_2(\hat{h}, c) \leqslant -\frac{1}{3}(\hat{h}_1^2 + \hat{h}_2^2)$ and hence

$$\begin{aligned} F_2^3(c)\, T_2(\hat{h}, c) \leqslant & -\left(\sum_{i=1}^{2} \hat{h}_i [g_{2,i}(c) - c_i] \right)^2 \\ & - [F_2^2(c)/3] \sum_{i=1}^{2} \hat{h}_i^2 \end{aligned}$$

from which it follows that $T_2(\hat{h}, c) \leqslant 0$.

Substatement 2.5. If Q is in Y_2, then there is a unique point y in D_2 such that $G_2(y) = Q$ and if $h_2(y) = (x_1, x_2)$, then

$$Q = g_{2,1}(x_1, x_2)\, \varphi_1 + g_{2,2}(x_1, x_2)\, \varphi_2.$$

Let

$$P = x_1 \varphi_1 + x_2 \varphi_2 - F_2(x_1, x_2)\, \varphi_3$$

and $P' = x_1' \varphi_1 + x_2' \varphi_2 - F_2(x_1', x_2')\, \varphi_3$ where (x_1', x_2') is in d_2 and distinct from (x_1, x_2). We shall show that

$$\|P - Q\|^2 < \|P' - Q\|^2$$

which rewritten is

$$0<(g_{2,1}(x_1,x_2)-x_1')^2+(g_{2,2}(x_1,x_2)-x_2')^2+F_2^2(x_1',x_2') \\ -(g_{2,1}(x_1,x_2)-x_1)^2-(g_{2,2}(x_1,x_2)-x_2)^2-F_2^2(x_1,x_2).$$

Note that for $1\leqslant i\leqslant 2$,

$$(x_i'-g_{2,i}(x_1,x_2))^2=(x_i'-x_i)^2+2(x_i'-x_i)(x_i-g_{2,i}(x_1,x_2)) \\ +(x_i-g_{2,i}(x_1,x_2))^2$$

and

$$g_{2,i}(x_1,x_2)-x_i=D_iF_2^2(x_1,x_i)/2.$$

Using the above we then have that

$$\|P'-Q\|^2-\|P-Q\|^2=F_2^2(x_1',x_2')+\sum_{i=1}^{2}(x_i'-x_i)^2 \\ -F_2^2(x_1,x_2)-\sum_{i=1}^{2}(x_i'-x_i)\,D_iF_2^2(x_1,x_i).$$

There is a number t in $(0,1)$ such that

$$F_2^2(x_1',x_2')=F_2^2(x_1,x_2)+\sum_{i=1}^{2}(x_i'-x_i)\,D_iF_2^2(x_1,x_2) \\ +\sum_{i=1}^{2}(x_i'-x_i)^2\,D_{i,i}F_2^2(c)/2 \\ +\sum_{i<j}^{2}(x_i'-x_i)(x_j'-x_j)\,D_{i,j}F_2^2(c),$$

where $c=(x_1+t(x_1'-x_1),x_2+t(x_2'-x_2))$ and notice that c is interior to d_2.

Let $\hat{h}_i=x_i'-x_i$ for $i=1,2$, then we have

$$\|P'-Q\|^2-\|P-Q\|^2=\sum_{i=1}^{2}\hat{h}_i^2D_{i,i}F_2^2(c)/2+\sum_{i<j}^{2}\hat{h}_i\hat{h}_jD_{i,j}F_2^2(c)+\sum_{i=1}^{2}\hat{h}_i^2 \\ =t_2(\hat{h},c)+\sum_{i=1}^{2}\hat{h}_i^2.$$

However from Lemma J we have that

$$t_2(\hat{h},c)+\sum_{i=1}^{2}\hat{h}_i^2\geqslant(\tfrac{1}{2})(\tfrac{1}{3})[\hat{h}_1^2+A_1L_1(c)\,\hat{h}_2^2]$$

from which it follows that

$$\|P' - Q\|^2 > \|P - Q\|^2.$$

Substatement 2.6. If W is in E_3^- and W is not in I_2, then W has a unique nearest point in S_2. If W is in I_2, then there is a point Q in Y_2 and a point P in S_2 such that W is in the interval $[Q, P]$ and P is the unique nearest point in S_2 to Q. Let C be the ball centered at W with radius $\|W - P\|$. If C contains a point of S_2 distinct from P, then P is not the unique nearest point in S_2 to Q which is a contradiction, hence P is the unique nearest point in S_2 to W.

Substatement 2.7 is clearly true.

Substatement 2.8. If W is in E_2^- and P is the unique nearest point in S_2 to W, then P is in E_2 because I_2 is convex. Hence P is in the boundary of D_2 and hence in S_1 because I_1 is convex and D_2 is convex. Therefore the unique nearest point to W in S_2 is in S_1 and is the unique nearest point in S_1 to W.

Substatement 2.9 is contained in the argument given for substatement 2.4.

Substatement 2.10. Recall that

$$JG_2 = D_1 g_{2,1} D_2 g_{2,2} - (D_1 g_{2,2})^2.$$

From Lemma G, $JG_2 = L_0 L_1 l(1, 2, 1)$ and there is a number $\bar{A}_2 > 0$ such that if $\bar{A}_2 > A_2 > 0$, then $l(1, 2, 1)$ is positive on d_2. If $J_2 = l(1, 2, 1)$, then $JG_2(x) \geqslant L_1(x) J_2(x)$ and $J_2(x) > 0$ for x in D_2.

Suppose that we have determined positive numbers $A_1^*, A_2^*, ..., A_{n-1}^*$ such that if $A_i^* > A_i > 0$ for $i = 1, 2, ..., n-1$ then statements 1 through n are correct.

Statement $n+1$. Substatement $n+1.1$. The set D_{n+1} is clearly closed and bounded. The set

$$d_{n+1}^- = \{(x_1, ..., x_{n+1}) : (x_1, ..., x_n) \in d_n, \ -F_n(x_1, ..., x_n) \leqslant x_{n+1} \leqslant 0\}$$

is closed and convex. Hence consider the set

$$d_{n+1}^+ = \{(x_1, ..., x_{n+1}) : (x_1, ..., x_n) \in d_n, 0 \leqslant x_{n+1} \leqslant a_n F_n(x_1, ..., x_n)\}.$$

If we can show that $a_n F_n$ is concave, i.e.,

$$2a_n F_n(x) \geqslant a_n F_n(x + \hat{h}) + a_n F_n(x - \hat{h}),$$

where $x = (x_1, ..., x_n)$, $\hat{h} = (\hat{h}_1, ..., \hat{h}_n)$, and each of x, $x + \hat{h}$, and $x - \hat{h}$ is in the interior of d_n, then d_{n+1}^+ is convex.

There are points c and c' between x and $x+\hat{h}$, and x and $x-\hat{h}$, respectively, such that

$$a_n F_n(x+\hat{h}) = a_n F_n(x) + \sum_{i=1}^{n} \hat{h}_i D_i a_n F_n(x) + \sum_{i=1}^{n} \hat{h}_i^2 D_{i,i} a_n F_n(c)/2 + \sum_{i<j}^{n} \hat{h}_i \hat{h}_j D_{i,j} a_n F_n(c)$$

and

$$a_n F_n(x-\hat{h}) = a_n F_n(x) - \sum_{i=1}^{n} \hat{h}_i D_i a_n F_n(x) + \sum_{i=1}^{n} \hat{h}_i^2 D_{i,i} a_n F_n(c')/2 + \sum_{i<j}^{n} \hat{h}_i \hat{h}_j D_{i,j} a_n F_n(c')$$

and thus we have that

$$a_n F_n(x+\hat{h}) + a_n F_n(x-\hat{h}) - 2a_n F_n(x) = [V_n(\hat{h}, c) + V_n(-\hat{h}, c')]/2,$$

where

$$V_n(\hat{h}, c) = \sum_{i=1}^{n} \hat{h}_i^2 D_{i,i} a_n F_n(c) + 2 \sum_{i<j}^{n} \hat{h}_i \hat{h}_j D_{i,j} a_n F_n(c).$$

If we can show that $0 \geqslant V_n(\hat{h}, c)$ when c is in the interior of d_n, then we will have that $a_n F_n$ is concave on d_n.

A series of computations and the definition of $T_n(\hat{h}, c)$ yields

$$\begin{aligned} V_n(\hat{h}, c) = a_n(c)\, T_n(\hat{h}, c) &+ A_n \Bigg\{ F_n(c) \left[\sum_{i=1}^{n} \hat{h}_i^2 D_{i,i} L_n(c) + 2 \sum_{i<j}^{n} \hat{h}_i \hat{h}_j D_{i,j} L_n(c) \right] \\ &+ 2 \sum_{i=1}^{n} \hat{h}_i^2 (D_i L_n(c))(D_i F_n(c)) \\ &+ 2 \sum_{i<j}^{n} \hat{h}_i \hat{h}_j [(D_j L_n(c))(D_i F_n(c)) + (D_i L_n(c))(D_j F_n(c))] \Bigg\}. \end{aligned}$$

In Lemmas E and F it is shown that there is a number $B_n > 0$ such that if $1 \leqslant i, j \leqslant n$, and c is in the interior of d_n, then

$$|D_{i,j} L_n(c)| \leqslant B_n$$

and

$$|D_i F_n(c)| \leqslant B_n / F_n(c).$$

Using the above we have that

$$V_n(\hat{h}, c) \leqslant a_n(c)\, T_n(\hat{h}, c) + A_n B_n \left\{ F_n(c) \left[\sum_{i=1}^{n} \hat{h}_i^2 + 2 \sum_{i<j}^{n} |\hat{h}_i \hat{h}_j| \right] + 2B_n \sum_{i=1}^{n} \hat{h}_i^2 / F_n(c) + 2B_n \sum_{i<j}^{n} |\hat{h}_i \hat{h}_j|\, 2/F_n(c) \right\};$$

$$V_n(\hat{h}, c)\, F_n^3(c) \leqslant a_n(c)\, T_n(\hat{h}, c)\, F_n^3(c) + A_n B_n F_n^2(c) \left\{ F_n^2(c) \left(\sum_{i=1}^{n} |\hat{h}_i| \right)^2 + 2B_n \left(\sum_{i=1}^{n} |\hat{h}_i| \right)^2 \right\}.$$

Using our bound for $T_n(\hat{h}, c)\, F_n^3(c)$, we then have that

$$V_n(\hat{h}, c)\, F_n^3(c) \leqslant -a_n(c) \left[\sum_{i=1}^{n} \hat{h}_i (g_{n,i}(c) - c_i) \right]^2 + F_n^2(c) \left(\sum_{i=1}^{n} \hat{h}_i^2 \right) \{ -a_n(c)/3^{-n} + nA_n B_n [F_n^2(c) + 2B_n] \}.$$

There is a number $A_n' > 0$ such that if $A_n' > A_n > 0$, then $nA_n B_n (F_n^2(c) + 2B_n) - a_n(c)\, 3^{-n} < 3^{-n}/2$ and hence

$$V_n(\hat{h}, c)\, F_n^3(c) \leqslant -a_n(c) \left[\sum_{i=1}^{n} \hat{h}_i (g_{n,i}(c) - c_i) \right]^2 - [F_n^2(c)\, 3^{-n}/2] \left(\sum_{i=1}^{n} \hat{h}_i^2 \right),$$

from which it follows that $V_n(\hat{h}, c) \leqslant 0$ on the interior of d_{n+1}. Since $d_{n+1} = d_{n+1}^+ \cup d_{n+1}^-$ and $d_{n+1}^+ \cap d_{n+1}^- = d_n$, the projection of d_{n+1}^+ onto d_n is the projection of d_{n+1}^- onto d_n is d_n, it follows that D_{n+1} is a bounded, closed, and convex set.

Substatement $n+1.2$ is clearly true.

Substatement $n+1.3$. We shall use Lemma A as found in Section 3. Let us first show that G_{n+1} restricted to the boundary of D_{n+1} is reversible.

If $(x_1, \dots, x_n)$ is in d_n, $h_n(x) = (x_1, \dots, x_n)$, $x_{n+1} = -F_n(x_1, \dots, x_n)$, and $h_{n+1}(y) = (x_1, \dots, x_n, x_{n+1})$, then $G_{n+1}(y) = G_n(x)$ and thus G_{n+1} is invertible on that subset of the boundary of D_{n+1} that is homeomorphic under h_{n+1} to

$$d_{n+1}^- = \{(x_1, \dots, x_{n+1}) : (x_1, \dots, x_n) \in d_n, \ -F_n(x_1, \dots, x_n) \leqslant x_{n+1} \leqslant 0\}.$$

If x and y are such that $h_n(x) = (x_1, \dots, x_n)$ is in d_n and $h_{n+1}(y) = (h_n(x), a_n F_n(h_n(x)))$, then

$$G_{n+1}(y) = \sum_{i=1}^{n} \{x_i + F_n^2(h_n(x))\, D_i a_n(h_n(x)) + a_n(h_n(x))[g_{n,i}(h_n(x)) - x_i]\}\, \phi_i + [a_n(h_n(x)) - 1]\, F_n(h_n(x))\, \phi_{n+1}.$$

For $1 \leqslant i \leqslant n$ let

$$k_{n,i}(h_n(x)) = F_n^2(h_n(x))\, D_i L_n(h_n(x)) + L_n(h_n(x))[g_{n,i}(h_n(x)) - x_i],$$
$$l_{n,i}(h_n(x)) = g_{n,i}(h_n(x)) + A_n k_{n,i}(h_n(x)),$$

and

$$H_n(x_1, \dots, x_n) = \sum_{i=1}^{n} [g_{n,i}(x_1, \dots, x_n) + A_n k_{n,i}(x, \dots, x_n)]\, \varphi_i,$$

then

$$G_{n+1}(y) = H_n(h_n(x)) + (a_n(h_n(x)) - 1)\, F_n(h_n(x))\, \varphi_{n+1}.$$

Notice that if $(x_1, \dots, x_n)$ is in the boundary of d_n, then $L_n(x_1, \dots, x_n) = 0$ and hence $F_n(x_1, \dots, x_n) = 0$ and thus $k_{n,i}(x_1, \dots, x_n) = 0$, from which it follows that H_n is reversible on the boundary of d_n.

We shall now show that there is a number $A_n'' > 0$ such that if $A_n'' > A_n > 0$, then JH_n, the Jacobian of H_n, is nonzero on the interior of d_n:

$$\begin{aligned} JH_n &= \det(D_i g_{n,j} + A_n D_i k_{n,j}) \qquad 1 \leqslant i, \quad j \leqslant n, \\ &= \binom{n}{0} JG_n + \sum_{t=1}^{n-1} A_n^t \binom{n}{t} D(n, t) + A_n^n JK_n, \end{aligned}$$

where JK_n is the Jacobian of $K_n = \sum_{i=1}^{n} k_{n,i} \varphi_i$, and $D(n, t)$ is a determinant having exactly t rows of the form $D_i g_{n,j}$ and $n - t$ rows of the form $D_i k_{n,j}$.

Recall that

$$JG_n = \left(\prod_{i=1}^{n-1} L_i \right) J_n,$$

where J_n is positive on D and also note that

$$JK_n = \sum_{(j_1, \ldots, j_n)} \pm D_1 k_{n,j_1} D_2 k_n, j_2 \cdots D_n k_{n,j_n},$$

where the sum is over all permutations $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$, and finally that

$$\prod_{i=1}^{n} D_i k_{n,j_i} = \prod_{i=1}^{n} D_i(F_n^2[D_{j_i} L_n] + (L_n/2)[D_{j_i} F_n^2])$$
$$= \prod_{i=1}^{n} ((3/2)\, m_n [D_i L_n](D_{j_i} L_n) + L_n C(i, j_i, n)),$$

where $m_n = 2/(a_{n-1} + 1)$ and

$$C(i, j_i, n) = [(D_i m_n)(D_{j_i} L_n) + m_n D_i(D_{j_i} L_n)](3/2)$$
$$+ (D_{j_i} m_n)(D_i L_n) + (L_n/2)\, D_i(D_{j_i} m_n)$$

which is bounded on d_n.

From Lemma C, for $1 \leqslant i \leqslant n$,

$$D_i L_n = \sqrt{L_{i-1}}\, B(i, n),$$

where $B(i, n)$ is bounded on d_n. Hence

$$\prod_{i=1} D_i k_{n,j_i} = \prod_{i=1} [(3/2)\, m_n \sqrt{L_{i-1} L_{j_i-1}}\, B(i, n)\, B(j_i, n) + L_n C(i, j_i, n)].$$

In the proof of Lemma E it is shown that for $1 \leqslant i, j_i \leqslant n$

$$L_n = \sqrt{L_{i-1} L_{j_i-1}}\, D(i, j_i, n),$$

where $D(i, j_i, n) = A(n, i)\, A(n, j_i)$ is bounded on d_n. For $1 \leqslant i, j_i \leqslant n$ let

$$E(i, j_i, n) = \tfrac{3}{2} m_n B(i, n)\, B(j_i, n) + C(i, j_i, n)\, D(i, j_i, n).$$

Then $D_i k_{n,j_i} = \sqrt{L_{i-1} L_{j_i-1}}\, E(i, j_i, n)$ and

$$\prod_{i=1}^{n} D_i k_{n,j_i} = \prod_{i=1}^{n} \sqrt{L_{i-1} L_{j_i-1}}\, E(i, j_i, n)$$
$$= \left(\prod_{i=0}^{n-1} L_i\right) E(j_1, j_2, \ldots, j_n),$$

where $E(j_1, j_2, ..., j_n) = \prod_{i=1}^n E(i, j_i, n)$, from which it follows that

$$|JK_n| \leqslant E(n) \prod_{i=1}^{n-1} L_i,$$

where $E(n) = \sum_{(j_1, ..., j_n)} |E(j_1, ..., j_n)|$.

Let us now consider $D(n, t)$, $1 \leqslant t \leqslant n-1$:

$$D(n, t) = \sum_{(j_1, ..., j_n)} \pm (D_1 H_{n,j_1} \cdots D_n H_{n,j_n}),$$

where $H_{n,j_i} = g_{n,j_i}$ or k_{n,j_i}.

From Lemmas E and F, for $1 \leqslant i, j_i \leqslant n$,

$$D_i g_{n,j_i} = \sqrt{L_{i-1} L_{j_i - 1}}\, B(i, n, j_i),$$

where $B(i, n, j_i)$ is bounded on d_n, and thus for $1 \leqslant i, j_i \leqslant n$,

$$D_i H_{n,j_i} = \sqrt{L_{i-1} L_{j_i - 1}}\, H(i, j_i, n),$$

where $H(i, j_i, n)$ is bounded on d_n. Therefore,

$$|D(n, t)| \leqslant \left(\prod_{i=1}^{n-1} L_i \right) H(n, t),$$

where $H(n, t)$ is bounded on d_n.

Combining all of the above we have then that

$$\begin{aligned} JH_n &= JG_n + \sum_{i=1}^{n-1} A_n^t \binom{n}{t} D(n, t) + A_n^n JK_n \\ &\geqslant \left(\prod_{i=1}^{n-1} L_i \right) \left[J_n - \sum_{t=1}^{n-1} \binom{n}{t} A_n^t\, |H(n, t)| - A_n^n E(n) \right], \end{aligned}$$

where J_n is positive on d_n. Hence there is a number $A_n'' > 0$ such that if $A_n'' > A_n > 0$, then

$$J_n - \sum_{t=1}^{n-1} \binom{n}{t} A_n^t\, |H(n, t)| - A_n^n E(n) > 0$$

on d_n, and thus $JH_n > 0$ on the interior of d_n, since $\prod_{i=1}^{n-1} L_i$ is positive on the interior of d_n. Thus we have that H_n is invertible on the boundary of d_n and has a local inverse at each interior point of d_n. Hence using Lemma A, H_n is a homeomorphism on d_n and thus G_{n+1} is invertible on the boundary of D_{n+1}.

We need to show that G_{n+1} has a local inverse at each point of the interior of D_{n+1}. To show this we shall show that JG_{n+1}, the Jacobian of G_{n+1}, is not zero on the interior of D_{n+1}.

For this portion we shall adopt the following notation:

$$d(i, k, j) = D_i g_{k,j} \qquad \text{for} \quad 1 \leqslant i, \quad j \leqslant k.$$

Thus we have that

$$JG_{n+1} = \det(d(i, n+1, j))$$

and, recalling that $d(i, n, j) = d(j, n, i)$, we expand along the main diagonal to find that

$$\begin{aligned}
JG_{n+1} &= ((n+1)!/2) \prod_{i=1}^{n+1} d(i, n+1, i) \\
&\quad - (n-1)! \sum_{v=1}^{n} \sum_{u=v+1}^{n+1} d^2(u, n+1, v) \prod_{\substack{k=1 \\ k \neq u,v}}^{n+1} d(k, n+1, k) \\
&= d(n+1, n+1, n+1)[(n-1)!] \Bigg[(n(n+1)/2) \prod_{i=1}^{n} d(i, n+1, i) \\
&\quad - \sum_{v=1}^{n-1} \sum_{u=v+1}^{n} d^2(u, n+1, v) \prod_{\substack{k=1 \\ k \neq u,v}}^{n} d(k, n+1, k) \\
&\quad - \sum_{v=1}^{n} (d^2(n+1, n+1, v)/d(n+1, n+1, n+1)) \prod_{\substack{k=1 \\ k \neq v}}^{n} d(k, n+1, k) \Bigg] \\
&= d(n+1, n+1, n+1)[(n-1)!]\, K(n+1).
\end{aligned}$$

Notice that

$$d(n+1, n+1, n+1) = A_n L_n/(a_n + 1),$$

which is positive on the interior of d_{n+1}, and that

$$\begin{aligned}
&[(n(n+1)/2)] \prod_{k=1}^{n} d(k, n+1, k) \\
&\qquad - \sum_{v=1}^{n-1} \sum_{u=v+1}^{n} d^2(u, n+1, v) \prod_{\substack{k=1 \\ k \neq u,v}}^{n} d(k, n+1, k)
\end{aligned}$$

$$= n \prod_{k=1}^{n} d(k, n+1, k) + \sum_{v=1}^{n-1} \sum_{u=v+1}^{n} \left[\prod_{k=1}^{n} d(k, n+1, k) \right.$$

$$\left. - d^2(u, n+1, v) \prod_{\substack{k=1 \\ k \neq u,v}}^{n} d(k, n+1, k) \right]$$

$$= n \prod_{k=1}^{n} d(k, n+1, k) + \sum_{v+1}^{n-1} \sum_{u=v+1}^{n} \left[\prod_{\substack{k=1 \\ k \neq u,v}}^{n} d(k, n+1, k) \right]$$

$$\times [d(u, n+1, u)\, d(v, n+1, v) - d^2(u, n+1, v)].$$

We now have that

$$K(n+1) = n \prod_{k=1}^{n} d(k, n+1, k) + \sum_{v=1}^{n-1} \sum_{u=v+1}^{n} \left[\prod_{\substack{k=1 \\ k \neq u,v}}^{n} d(k, n+1, k) \right]$$

$$\times [d(u, n+1, u)\, d(v, n+1, v) - d^2(u, n+1, v)]$$

$$- \sum_{v=1}^{n} [d^2(n+1, n+1, v)/d(n+1, n+1, n+1)] \prod_{\substack{k=1 \\ k \neq v}}^{n} d(k, n+1, k)$$

$$= \sum_{v=1}^{n-1} \sum_{u=v+1}^{n} \left[\prod_{\substack{k=1 \\ k \neq u,v}}^{n} d(k, n+1, k) \right]$$

$$\times [d(u, n+1, u)\, d(v, n+1, v) - d^2(u, n+1, v)]$$

$$+ \sum_{v=1}^{n} \left[\prod_{\substack{k=1 \\ k \neq v}}^{n} d(k, n+1, k) \right] [d(v, n+1, v)\, d(n+1, n+1, n+1)$$

$$- d^2(n+1, n+1, v)]/d(n+1, n+1, n+1).$$

From Lemmas F and G we have that there is a number $A_n''' > 0$ such that if $A_n''' > A_n > 0$, then $D_u g_{n+1,u} D_v g_{n+1,v} - (D_u g_{n+1,v})^2 > 0$ on the interior of d_{n+1} for $1 \leqslant u \leqslant n+1$, and $D_k g_{n+1,k} = L_{k-1} B(k, n+1, k) > 0$ for $1 \leqslant k \leqslant n$.

Hence $K(n+1)$ is positive on the interior of D_{n+1} and thus JG_{n+1} is nonzero on the interior of D_{n+1}. Since G_{n+1} has a local inverse at each interior point of D_{n+1}, Lemma A applies, and thus G_{n+1} is a homeomorphism.

A few additional remarks at this point will establish $n+1.10$. In Lemma G it is shown that for $n = 2, 3, \ldots$, $1 \leqslant u$, $v \leqslant n$,

$$d(u, n, u)\, d(v, n, v) - d^2(u, n, v) = L_{u-1} L_{v-1} l(u, n, v),$$

where $l(u, n, v)$ is positive on d_n and for $1 \leqslant k \leqslant n$, and from Lemma F we have that

$$d(k, n, k) = L_{k-1} B(k, n, k),$$

where $B(k, n, k)$ is positive on d_n. Hence

$$\begin{aligned} JG_{n+1} = {} & L_n B(n+1, n+1, n+1)(n-1)! \left\{ \sum_{v=1}^{n} \sum_{u=v+1}^{n} \left[\prod_{\substack{k=1 \\ k \neq u,v}}^{n} L_{k-1} \right] \right. \\ & \left. \times \left[\prod_{\substack{k=1 \\ k \neq u,v}}^{n} B(k, n+1, k) \right] [L_{u-1} L_{v-1} l(u, n+1, v)] \right\} \\ & + [(1/d(n+1, n+1, n+1))] \sum_{v=1}^{n} \left[\prod_{\substack{k=1 \\ k \neq v}}^{n} L_{k-1} \right] \\ & \times \left[\prod_{\substack{k=1 \\ k \neq v}}^{n} B(k, n+1, k) \right] [L_n L_{v-1} l(n+1, n+1, v)] \\ = {} & \left(\prod_{i=1}^{n} L_i \right) J_{n+1}, \end{aligned}$$

where $J_{n+1} > 0$ on D_{n+1}.

Substatement $n+1.4$. To show that I_{n+1}, the region bounded by S_{n+1} and D_{n+1}, is convex it is sufficient to show that F_{n+1} is concave, i.e., that if $x = (x_1, ..., x_{n+1})$, $\hat{h} = (\hat{h}_1, ..., \hat{h}_{n+1})$, and each of x, $x + \hat{h}$, and $x - \hat{h}$ is in the interior of d_{n+1}, then

$$2F_{n+1}(x) \geqslant F_{n+1}(x + \hat{h}) + F_{n+1}(x - \hat{h}).$$

There are points c and c' between x and $x + \hat{h}$, and x and $x - \hat{h}$, respectively, such that

$$\begin{aligned} F_{n+1}(x + \hat{h}) = {} & F_{n+1}(x) + \sum_{i=1}^{n+1} \hat{h}_i D_i F_{n+1}(x) + \sum_{i=1}^{n+1} \hat{h}_i^2 D_{i,i} F_{n+1}(c)/2 \\ & + \sum_{i<j}^{n+1} \hat{h}_i \hat{h}_j D_{i,j} F_{n+1}(c) \end{aligned}$$

and

$$\begin{aligned} F_{n+1}(x - \hat{h}) = {} & F_{n+1}(x) - \sum_{i=1}^{n+1} \hat{h}_i D_i F_{n+1}(x) + \sum_{i=1}^{n+1} \hat{h}_i^2 D_{i,i} F_{n+1}(c')/2 \\ & + \sum_{i<j}^{n+1} \hat{h}_i \hat{h}_j D_{i,j} F_{n+1}(c'). \end{aligned}$$

Using the above we then have that

$$F_{n+1}(x+\hat{h})+F_{n+1}(x-\hat{h})-2F_{n+1}(x)=(T_{n+1}(\hat{h},c)+T_{n+1}(-\hat{h},c'))/2,$$

where

$$T_{n+1}(\hat{h},c)=\sum_{i=1}^{n+1}\hat{h}_i^2 D_{i,i}F_{n+1}(c)+2\sum_{i<j}^{n+1}\hat{h}_i\hat{h}_j D_{i,j}F_{n+1}(c)$$

for c, $c+\hat{h}$, and $c-\hat{h}$ in the interior of d_{n+1}. For $1\leqslant i, j\leqslant n+1$,

$$D_{i,j}F_{n+1}=(D_{i,j}F_{n+1}^2)/2F_{n+1}-(D_iF_{n+1}^2)(D_jF_{n+1}^2)/4F_{n+1}^3,$$

$$D_iF_{n+1}^2(x)/2=g_{n+1,i}(x)-x_i,$$

$$D_{i,i}F_{n+1}^2(x)/2=D_i\,g_{n+1,i}(x)-1$$

and if $i\neq j$,

$$D_{i,j}F_{n+1}^2/2=D_i\,g_{n+1,j}.$$

We now have that

$$T_{n+1}(\hat{h},c)\,F_{n+1}^3(c)=-\left[\sum_{i=1}^{n+1}\hat{h}_i(g_{n+1,i}(c)-c_i)\right]^2+F_{n+1}^2(c)\,t_{n+1}(\hat{h},c),$$

where

$$t_{n+1}(\hat{h},c)=-\sum_{i=1}^{n+1}\hat{h}_i^2(1-D_i\,g_{n+1,i}(c))+2\sum_{i=1}^{n}\hat{h}_i\sum_{j=i+1}^{n+1}\hat{h}_jD_i\,g_{n+1,j}(c)$$

for each of c, $c+\hat{h}$, and $c-\hat{h}$ in the interior of d_{n+1}.

By Lemma I we have that $t_{n+1}(\hat{h},c)\leqslant -3^{-n}\sum_{i=1}^{n+1}\hat{h}_i^2$ and thus

$$T_{n+1}(\hat{h},c)\,F_{n+1}^2(c)\leqslant -[F_{n+1}^2(c)\,3^{-n}]\sum_{i=1}^{n+1}\hat{h}_i^2-\left[\sum_{i=1}^{n+1}\hat{h}_i(g_{n+1,i}(c)-c_i)\right]^2$$

and therefore I_{n+1} is convex.

Notice that we have also established $n+1.9$.

Substatement $n+1.5$. If Q is in Y_{n+1}, then there is a unique point X in D_{n+1} such that $G_{n+1}(X)=Q$. Recall that

$$G_{n+1}(X)=\sum_{i=1}^{n+1}g_{n+1,i}(h_{n+1}(X))\,\varphi_i.$$

Let

$$h_{n+1}(X) = x = (x_1, \ldots, x_{n+1}),$$

$$x' = (x_1', \ldots, x_{n+1}') \text{ be a point in } d_{n+1} \text{ distinct from } x,$$

$$P = \sum_{i=1}^{n+1} x_i \varphi_i - F_{n+1}(x)\, \varphi_{n+2},$$

and

$$P' = \sum_{i=1}^{n+1} x_i' \varphi_i - F_{n+1}(x')\, \varphi_{n+2}.$$

Then

$$\|P - Q\|^2 = \sum_{i=1}^{n+1} (g_{n+1,i}(x) - x_i)^2 + F_{n+1}^2(x)$$

and

$$\|P' - Q\|^2 = \sum_{i=1}^{n+1} (g_{n+1,i}(x) - x_i')^2 + F_{n+1}^2(x').$$

Note that for $1 \leqslant i \leqslant n+1$,

$$(g_{n+1,i}(x) - x_i')^2 = (x_i - x_i')^2 + 2(x_i - x_i')(g_{n+1,i}(x) - x_i) + (g_{n+1,i}(x) - x_i)^2,$$

and thus we have that

$$\|P' - Q\|^2 - \|P - Q\|^2 = \sum_{i=1}^{n+1} [(x_i - x_i')^2 + 2(x_i - x_i')(g_{n+1,i}(x) - x_i)] + F_{n+1}^2(x') - F_{n+1}^2(x).$$

There is a point c between x and x' such that

$$F_{n+1}^2(x') = F_{n+1}^2(x) + \sum_{i=1}^{n+1} \hat{h}_i D_i F_{n+1}^2(x) + \sum_{i=1}^{n+1} \hat{h}_i^2 D_{i,i} F_{n+1}^2(c)/2 + \sum_{i<j}^{n+1} \hat{h}_i h_j D_{i,j} F_{n+1}^2(c)$$

where $\hat{h}_i = x_i' - x_i$ for $1 \leqslant i \leqslant n+1$. Note that for $1 \leqslant i \leqslant n+1$,

$$g_{n+1,i}(x) - x_i = D_i F_{n+1}^2(x)/2,$$

and hence we have that

$$\begin{aligned}\|P' - Q\|^2 - \|P - Q\|^2 &= \sum_{i=1}^{n+1} \hat{h}_i^2 + \sum_{i=1}^{n+1} \hat{h}_i^2 D_{i,i} F_{n+1}^2(c)/2 \\ &\quad + \sum_{i<j}^{n+1} \hat{h}_i \hat{h}_j D_{i,j} F_{n+1}^2(c) \\ &= \sum_{i=1}^{n+1} \hat{h}_i^2 - \sum_{i=1}^{n+1} \hat{h}_i^2 [1 - D_i g_{n+1,i}(c)] \\ &\quad + 2 \sum_{i<j}^{n+1} \hat{h}_i \hat{h}_j D_i g_{n+1,j}(c) \\ &= \sum_{i=1}^{n+1} \hat{h}_i^2 + t_{n+1}(\hat{h}, c).\end{aligned}$$

Now by Lemma J, $t_{n+1}(\hat{h}, c) + \sum_{i=1}^{n+1} \hat{h}_i^2 > 0$ provided each of c, $c + \hat{h}$, and $c - \hat{h}$ are interior to d_{n+1}. Hence

$$\|P' - Q\|^2 > \|P - Q\|^2.$$

Since G_{n+1} is a homeomorphism, it follows that there is a 1–1 correspondence between Y_{n+1} and S_{n+1}.

Substatement $n+1.6$. If W is in E_{n+2}^- and W is not in I_{n+1}, then W has a unique nearest point in S_{n+1}. If W is in I_{n+1}, then there is a point Q in Y_{n+1} and a point P in S_{n+1} such that P is the unique nearest point in S_{n+1} to Q and W is in $[Q, P]$. Let C be the ball centered at W with radius $\|W - P\|$. If C contains a point of S_{n+1} distinct from P, then such a point would be closer to Q than P is to Q, which is a contradiction; hence P is the unique nearest point in S_{n+1} to W.

Substatement $n+1.7$. Clearly $S_n \subseteq S_{n+1}$.

Substatement $n+1.9$. This argument is contained in the argument for substatement $n+1.4$.

Substatement $n+1.10$. This argument is contained in the argument for substatement $n+1.3$.

3

LEMMA A. *Suppose K is a closed, bounded, and convex subset of E_n, that has an interior point. Let f be a continuous function from K into E_n such that f restricted to the boundary of K is reversible and each interior point of K is* *contained in an open subset of K such that f restricted to this open subset is a homeomorphism, i.e., f has a local inverse at each interior point. Then f is a homeomorphism.*

Proof. Since the boundary of K is homeomorphic to S^{n-1} and f, when restricted to the boundary of K, is a homeomorphism, it follows that the image of the boundary of K separates E_n. Hence the image of the boundary of K does not intersect the image of the interior of K. Let L be the set to which x belongs only in case x is in K and there is a point y in K, distinct from x, such that $f(x) = f(y)$. It is clear that L is closed and hence compact and also that L contains no point of the boundary of K. There is a point p in the boundary of K and a point q in L such that $\|p - q\|$ is the distance from L to the boundary of K. Let q' be a point in L distinct from q such that $f(q') = f(q)$. Let R and R' be two circular regions centered at q and q', respectively, such that the sum of their radii is less than $\|q - q'\|/3$, and f restricted to each of R and R' is a homeomorphism. Since $f(q) = f(q')$, it follows that $f(R) \cap f(R')$ exists and thus each point of R that has an image in $f(R')$ is also in L. Thus there is a point x in the open interval (q, p) that must belong to L, and therefore is closer to p than q is to p, which is a contradiction. Hence no two points of K have the same image under f and thus f is a homeomorphism on K.

LEMMA B. *If x is in d_1, then*

$$0 \leqslant L_1(x) \leqslant (3/2)^2,$$
$$F_1^2(x) \leqslant (3/2),$$

and if $0 < A_1 \leqslant (2/3)^2$, *then*

$$1 \leqslant a_1(x) \leqslant 2.$$

Moreover, if for each positive integer n,

$$0 < A_{n+1} \leqslant (2/3)^{2n-1},$$

then for $m = 1, 2, \ldots$ *and* $x = (x_1, \ldots, x_{m+1})$ *is in* d_{m+1}

$$0 \leqslant L_{m+1}(x) \leqslant (3/2)^{2m-1},$$
$$F_{m+1}^2(x) \leqslant (3/2)^{2m-1},$$

and $1 \leqslant a_{m+1}(x) \leqslant 2$.

The proof is a straightforward induction argument and omitted here.

LEMMA C. *If n is a positive integer and t is a positive integer not exceeding n, then $D_t L_n = L_{t-1}^{1/2} B(n, t)$, where $B(n, t)$ is bounded on d_n.*

Proof. We shall use the following notation:

$$L_{n+1} = a_n F_n^2 + (a_n - 1) F_n x_{n+1} - x_{n+1}^2.$$

Then $D_1 L_1 = (1 - 2x_1) = L_0^{1/2} B(1, 1)$, and

$$D_1 L_2 = A_1(D_1 L_1) F_1^2 + 2a_1(D_1 F_1^2/2) + A_1(D_1 L_1) F_1 x_2 + (a_1 - 1)(D_1 F_1^2/2)(x_2/F_1)$$

$$D_1 L_2 = L_0^{1/2} B(1, 2),$$

and finally

$$\begin{aligned} D_2 L_2 &= (a_1 - 1) F_1 - 2x_2 \\ &= F_1[(a_1 - 1) - 2(x_2/F_1)] \\ &= L_1^{1/2}[(a_1 - 1) - 2(x_2/F_1)](2/3)^{1/2} \end{aligned}$$

$$D_2 L_2 = L_1^{1/2} B(2, 2).$$

Suppose now that n is a positive integer such that if $1 \leqslant l \leqslant n$ and $1 \leqslant t \leqslant l$, then $D_t L_l = L_{t-1}^{1/2} B(l, t)$ where $B(l, t)$ is bounded on d_l.

Consider now

$$\begin{aligned} D_{n+1} L_{n+1} &= F_n[(a_n - 1) - 2(x_{n+1}/F_n)] \\ &= L_n^{1/2} B(n+1, n+1) \end{aligned}$$

and $B(n+1, n+1)$ is bounded on d_{n+1}.

Suppose that $1 \leqslant t < n+1$, then

$$\begin{aligned} D_t L_{n+1} = {} & A_n(D_t L_n) F_n^2 + 2a_n(D_t L_n)/(a_{n-1} + 1) \\ & - 2a_n L_n A_{n-1}(D_t L_{n-1})/(a_{n-1} + 1)^2 \\ & + A_n(D_t L_n) F_n x_{n+1} \\ & + (a_n - 1)(x_{n+1}/F_n)[(D_t L_n)/(a_{n-1} + 1) \\ & - L_n A_{n-1}(D_t L_{n-1})/(a_{n-1} + 1)^2] \\ = {} & L_{t-1}^{1/2}[B(n, t)[4a_n - 2 + 3(a_n - 1)(x_{n+1}/F_n)]/[a_{n-1} + 1] \\ & - B(n-1, t)[2a_n + (a_n - 1)(x_n + 1/F_n) A_{n-1} L_n]/[a_{n-1} + 1]^2] \\ = {} & L_{t-1}^{1/2} B(n+1, t). \end{aligned}$$

where it is to be understood that

$$B(i, j) = 0 \qquad \text{if} \quad i < j.$$

LEMMA D. *If n is a positive integer and $1 \leqslant k \leqslant n$, then*

$$g_{n,k} - x_k = L_{k-1}^{1/2} \bar{B}(n, k),$$

where $\bar{B}(n, k)$ is bounded on d_n.

Proof. If n is a positive integer and $1 \leqslant k \leqslant n$, then

$$g_{n,k} - x_k = D_k F_n^2/2.$$

If $k < n$, then

$$\begin{aligned} D_k F_n^2/2 &= [D_k L_n - L_n A_{n-1} D_k L_{n-1}/(a_{n-1}+1)]/(a_{n-1}+1) \\ &= L_{k-1}^{1/2} \bar{B}(n, k), \end{aligned}$$

where $\bar{B}(n, k)$ is bounded on d_n. If $k = n$, then a similar argument applies.

LEMMA E. *If n is a positive integer and $1 \leqslant i \leqslant k \leqslant n$ then $D_{i,k} L_n$ is bounded on d_n and $D_i g_{n,k} = L_{i-1}^{1/2} L_{k-1}^{1/2} B(i, n, k)$, where $B(i, n, k)$ is bounded on d_n.*

Proof.

$$\begin{aligned} D_1 g_{2,2} &= A_1(D_1 L_1) F_1(1 + x_2/F_1)/(a_1 + 1) \\ &\quad + A_1 L_1(D_1 F_1^2/2)/F_1(a_1 + 1) \\ &\quad - A_1^2 L_1(F_1 + x_2)(D_1 L_1)/(a_1 + 1)^2 \\ &= L_1^{1/2}[A_1(D_1 L_1)(1 + x_2/F_1)(2/3)^{1/2}/(a_1 + 1) \\ &\quad + A_1(3/2)^{1/2}(g_{1,1} - x_1)/(a_1 + 1) \\ &\quad - A_1^2 L_1^{1/2}(F_1 + x_2)(D_1 L_1)/(a_1 + 1)^2] \\ &= L_0^{1/2} L_1^{1/2} B(1, 2, 2), \end{aligned}$$

where $B(1, 2, 2)$ is bounded on d_2. Note that $L_0 \equiv 1$,

$$\begin{aligned} D_{1,2} L_2 &= D_1(D_2 L_2) \\ &= A_1(D_1 L_1)[(3/2) F_1] \end{aligned}$$

which is bounded on d_2.

Suppose now that n is a positive integer such that if $1 \leqslant l \leqslant n$ and $1 \leqslant i < k \leqslant l$, then $D_i g_{l,k} = L_{i-1}^{1/2} L_{k-1}^{1/2} B(i, l, k)$, where $B(i, l, k)$ is bounded on d_l, and $D_{i,k} L_l$ is bounded on d_l.

Suppose $1 \leqslant i < k \leqslant n+1$, then if $k < n+1$, we have that

$$\begin{aligned} D_i g_{n+1,k} = {} & A_n(D_{i,k} L_n)\, 2L_n((1 + x_{n+1}/F_n)/(a_n+1)^2)/(a_{n-1}+1) \\ & + A_n(D_k L_n)(1 + x_{n+1}/F_n)(D_i F_n^2/2)/(a_n+1)^2 \\ & - 2A_n^2(D_k L_n)((F_n + x_{n+1})/(a_n+1))^2(D_i L_n)/(a_n+1) \\ & + (D_i g_{n,k})(2a_n + (a_n - 1)\, x_{n+1}/F_n)/(a_n+1) \\ & + (g_{n,k} - x_k)[A_n(D_i L_n)(2 + x_{n+1}/F_n)/(a_n+1) \\ & - A_n L_n(x_{n+1}/F_n)(D_i F_n^2/2)/F_n^2(a_n+1) \\ & - A_n(2a_n + (a_n-1)(x_{n+1}/F_n)(D_i L_n)/(a_n+1)^2]. \end{aligned}$$

If t is a positive integer, then

$$\begin{aligned} L_{t+1} &= L_t 2[a_t + (a_t - 1)(x_{t+1}/F_t) - (x_{t+1}/F_t)^2]/(a_{t-1}+1) \\ &= L_t A^2(t-1) \end{aligned}$$

and hence if $1 \leqslant s \leqslant t$,

$$L_t^{1/2} = L_{t-s}^{1/2} \left[\prod_{r=1}^{s} A(t-r) \right] = L_{t-s}^{1/2} A(t, t-s+1)$$

from which it follows that for $0 \leqslant i-1 < n$,

$$L_n^{1/2} = L_{i-1}^{1/2} A(n, i).$$

Therefore,

$$D_i g_{n+1,k} = L_{i-1}^{1/2} L_{k-1}^{1/2} B(i, n+1, k)$$

where $B(i, n+1, k)$ is bounded on d_{n+1}.

A similar argument holds in the case $k = n+1$.

Consider now $D_{i,k} L_{n+1}$, i.e.,

$$\begin{aligned} D_{i,k} L_{n+1} = {} & (D_i g_{n+1,k})(a_n+1) + A_n D_i L_n(g_{n+1,k} - x_k) \\ & + (F_{n+1}^2/2)\, A_n(D_{i,k} L_n) + A_n D_k L_n(g_{n+1,i} - x_i) \end{aligned}$$

which is bounded on d_{n+1}.

LEMMA F. *There is a positive number sequence* $A_1, A_2, \ldots$ *such that for every positive integer n,*

$$0 < D_1 g_{n,1} \leqslant \sum_{t=1}^{n} (\tfrac{1}{3})^t < \tfrac{1}{2} \qquad on \quad d_n,$$

there is a number $B_n > 0$ *such that for* $1 \leqslant t \leqslant n$,

$$|D_{t,t} L_n| \leqslant B_n$$

and

$$D_t g_{n,t} = L_{t-1} B(t, n, t),$$

where $B(t, n, t)$ *is positive and bounded on* d_n.

Proof. First $D_1 g_{1,1} = \frac{1}{3}$, $D_{1,1} L_1 = -2$, $|D_{1,1} L_1| \leqslant B_1 = 2$, and $D_1 g_{1,1} = L_0 B(1, 1, 1)$ on d_1. Suppose now that $A_1, \ldots, A_{n-1}$ have been chosen such that for $1 \leqslant l \leqslant n$, $0 < D_1 g_{l,1} \leqslant \sum_{t=1}^{l} (\frac{1}{3})^t$ on d_l. Consider now $D_1 g_{n+1,1}$. Straightforward calculations yield

$$D_1 g_{n+1,1} = (2/(a_n + 1))\, D_1 g_{n,1} + A_n D(1, n+1, 1),$$

where $D(1, n+1, 1)$ is bounded on d_{n+1}.

Hence there is an $\bar{A}_n > 0$ such that

$$(2/(a_n + 1))\, D_1 g_{n,1} - \bar{A}_n\, |D(1, n+1, 1)| > D_1 g_{n,1}/(a_n + 1) > 0$$

on d_{n+1}.

Therefore if $\bar{A}_n > A_n > 0$, then $D_1 g_{n+1,1} > 0$ on d_{n+1}; moreover,

$$\begin{aligned} D_1 g_{n+1,1} &\leqslant (2/(a_n + 1))\, D_1 g_{n,1} + A_n\, |D(1, n+1, 1)| \\ &\leqslant D_1 g_{n,1} + A_n\, |D(1, n+1, 1)|. \end{aligned}$$

There is $\bar{\bar{A}}_n > 0$ such that $\bar{\bar{A}}_n\, |D(1, n+1, 1) < (\frac{1}{3})^{n+1}$ and hence

$$D_1 g_{n,1} + \bar{\bar{A}}_n\, |D(1, n+1, 1)| \leqslant \sum_{t=1}^{n} (\tfrac{1}{3})^t + (\tfrac{1}{3})^{n+1}.$$

Thus we have that on d_{n+1},

$$0 < D_1 g_{n+1,1} < \sum_{t=1}^{n+1} (\tfrac{1}{3})^t,$$

provided $0 < A_n \leqslant \min\{\bar{A}_n, \bar{\bar{A}}_n\}$.

Let us now turn to the second part of the lemma. Notice first that

$$D_2 g_{2,2} = L_1(A_1/(a_1+1)) = L_1 B(2,2,2),$$

where $A_1/(a_1+1)$ is positive on d_2.

Suppose that $A_1, A_2, ..., A_n$ have been chosen such that if $2 \leqslant t \leqslant l \leqslant n$ then there is a number B_l and a function $B(t, l, t)$ such that $|D_{t,t} L_l| \leqslant B_l$ on d_l and $D_t g_{l,t} = L_{t-1} B(t, l, t)$, where $B(t, l, t)$ is positive and bounded on d_l.

Suppose now that $2 \leqslant t \leqslant n+1$ and consider $D_t g_{n+1,t}$. There is one special case, namely when $t = n+1$, where $D_{n+1} g_{n+1,n+1} = L_n A_n/(a_n+1) = L_n B(n+1, n+1, n+1)$. It is clear that $B(n+1, n+1, n+1)$ is positive on d_{n+1}.

Consider now the remaining cases, i.e., $2 \leqslant t \leqslant n$. Straightforward calculations yield

$$D_t g_{n+1,t} = (2/(a_n+1))\, D_t g_{n,t} + A_n L_{t-1} D(t, n+1, t),$$

where $D(t, n+1, t)$ is bounded on d_{n+1}. Thus we have that

$$\begin{aligned} D_t g_{n+1,t} &= (2/(a_n+1))\, L_{t-1} B(t, n, t) + A_n L_{t-1} D(t, n+1, t) \\ &= L_{t-1}[(2/(a_n+1))\, B(t, n, t) + A_n D(t, n+1, t)], \end{aligned}$$

where $B(t, n, t)$ is positive on d_n.

Choose $\bar{\bar{\bar{A}}}_n > 0$ such that

$$B(t, n, t) + \bar{\bar{\bar{A}}}_n\, |D(t, n+1, t)| > 0 \qquad \text{on} \quad d_{n+1}.$$

Then if $\bar{\bar{\bar{A}}}_n > A_n > 0$,

$$(2/(a_n+1))\, B(t, n, t) + A_n D(t, n+1, t) > 0 \qquad \text{on} \quad d_{n+1}$$

and thus

$$D_t g_{n+1,t} = L_{t-1} B(t, n+1, t),$$

where $B(t, n+1, t)$ is bounded and positive on d_{n+1}.

Hence select $0 < A_n < \min\{\bar{A}_n, \bar{\bar{A}}_n, \bar{\bar{\bar{A}}}_n\}$. A straightforward computation shows that there is a positive number B_{n+1} such that $|D_{t,t} l_{n+1}| \leqslant B_{n+1}$ for $1 \leqslant t \leqslant n+1$ on d_l.

Notice that if $t \neq n+1$, then as A_n decreases, $B(t, n+1, t)$ increases.

Lemma G. *If $1 \leqslant u < v \leqslant n$, then there is a number $\bar{A}_n > 0$ such that if $\bar{A}_n > A_n > 0$, then $D_u g_{n,u} D_v g_{n,v} - (D_u g_{n,v})^2 = L_{u-1} L_{v-1} l(u, n, v)$, where $l(u, n, v)$ is positive on d_n.*

Proof. For $1 \leqslant u < v < n+1$,

$$D_u g_{n+1,u} = (2/(a_n+1))\, D_u g_{n,u} + A_n D(u, n+1, u),$$

and

$$D_u g_{n+1,v} = (2/(a_n+1))\, D_u g_{n,v} + A_n D(u, n+1, v).$$

Therefore,

$$\begin{aligned}(D_u g_{n+1,u} D_v g_{n+1,v} - (D_u g_{n+1,v})^2)\\ &= (2/(a_n+1))^2 [D_u g_{n,u} D_v g_{n,v} - (D_u g_{n,v})^2]\\ &\quad + A_n[(2/(a_n+1))[D_u g_{n,u} D(v, n+1, v) + D_v g_{n,u} D(u, n+1, u)\\ &\quad + A_n D(u, n+1, u)\, D(v, n+1, v)\\ &\quad - A_n D^2(u, n+1, v) - (4/(a_n+1))\, D_u g_{n,v} D(u, n+1, v)].\end{aligned}$$

Notice that

$$D(u, n+1, u) = L_{u-1} B(u, n+1, u)$$

and

$$D(u, n+1, v) = L_{u-1}^{1/2} L_{u-1}^{1/2} B(u, n+1, v).$$

Therefore,

$$\begin{aligned}D_u g_{n+1,u} D_v g_{n+1,v} - (D_u g_{n+1,v})^2\\ &= (2/(a_n+1))^2\, L_{u-1} L_{v-1}\, l(u, n, v)\\ &\quad + A_n L_{u-1} L_{v-1}[(2/(a_n+1))[B(u, n+1, u)\, B(v, n+1, u)\\ &\quad + B(v, n, v)\, B(u, n+1, u)]\\ &\quad + A_n[B(u, n+1, u)\, B(v, n+1, v)\\ &\quad - B^2(u, n+1, v) - (4/(a_n+1))\, B(u, n, v)\, \bar{B}(u, n+1, v)\\ &= L_{u-1} L_{v-1}[(2/(a_n+1))^2\, l(u, n, v)\\ &\quad + A_n C(u, n+1, v)],\end{aligned}$$

where $C(u, n+1, v)$ is bounded on d_{n+1}.

Hence choose $\bar{A}_n > 0$ such that

$$l(u, n, v) + \bar{A}_n\, |C(u, n+1, v)| > 0 \qquad \text{on} \quad d_{n+1}.$$

Therefore if $\bar{A}_n > A_n > 0$, then

$$D_u g_{n+1,u} D_v g_{n+1,v} - (D_u g_{n+1,v})^2 = L_{u-1} L_{v-1} l(u, n+1, v)$$

if $\bar{A}_n > A_n > 0$.

LEMMA H. *If* $1 \leqslant s < t \leqslant n$, *then*

$$D_s g_{n,t} = [(a_n+1)/2] \sum_{l=t}^{n} A_{l-1} D(s, l, t) \prod_{r=l}^{n} [2/(a_r+1)].$$

Proof. Proof follows by simple induction using

$$D_s g_{n,t} = [2/(a_{n-1}+1)]\, D_s g_{n-1,t} + A_{n-1} D(s, n, t).$$

LEMMA I. *There is a positive number sequence* $\{A_i\}_{i=1}^{\infty}$ *such that if n is an integer greater than* 2 *and c is in the interior of* d_n, *then*

$$-t_n(\hat{h}, c) = \sum_{i=1}^{n} \hat{h}_i^2[1 - D_i g_{n,i}(c)] - 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \hat{h}_i \hat{h}_j D_i g_{n,j}(c)$$
$$\geqslant 3^{1-n} \sum_{i=1}^{n} \hat{h}_i^2.$$

Proof. Suppose that c is in the interior of d_2, then

$$\begin{aligned}-t_2(\hat{h}, c) = {} & \sum_{i=1}^{2} \hat{h}_i^2/2 + \hat{h}_1^2[a_1 + 1 - 2D_1 g_{1,1} \\ & - 2A_1(a_1+1)\, D(1,2,1)]/[2(a_1+1)] \\ & + \hat{h}_2^2[a_1 + 1 - A_1 L_1]/[2(a_1+1)] - 2\hat{h}_1 \hat{h}_2 [A_1 D(1,2,2)] \\ \geqslant {} & (1/3) \sum_{i=1}^{2} \hat{h}_i^2 + \hat{h}_1^2[\tfrac{2}{3} + A_1[L_1 - 2(a_1+1)\, D(1,2,1)]/[2(a+1)] \\ & + \hat{h}_2^2[2 - A_1 L_1]/[2(a_1+1)] - 2\hat{h}_1 \hat{h}_2 [A_1 D(1,2,2)].\end{aligned}$$

Let us require that A_1 be chosen such that

(1) $\frac{2}{3} + A_1[L_1 - 2(a_1+1)\, D(1,2,1) > \frac{1}{2}$ and

(2) $[2 - A_1 L_1] > \frac{1}{2}$ on d_2, then

$$\begin{aligned}-t_2(\hat{h}, c) \geqslant (\tfrac{1}{3}) \sum_{i=1}^{2} \hat{h}_i^2 + [\hat{h}_1^2 - 2\hat{h}_1 \hat{h}_2 A_1 [4(a_1+1)\, D(1,2,2)] \\ + \hat{h}_2^2 A_1^2 [4(a_1+1)\, B(1,2,2)]^2]/[4(a_1+1)].\end{aligned}$$

Let us now, in addition, require that A_1 be chosen such that

$$1-16A_1^2(a_1+1)^2\,D^2(1,2,2)>0 \qquad \text{on} \quad d_2,$$

from which it follows that

$$-t_2(\hat{h},c)\geqslant \tfrac{1}{3}\sum_{i=1}^{2}\hat{h}_i^2.$$

Suppose now that $A_1, A_2, ..., A_n$ have been chosen such that if $2\leqslant m\leqslant n$ and c is the interior of d_m, then

$$-t_m(\hat{h},c)\geqslant (\tfrac{1}{3})^{m-1}\sum_{i=1}^{m}\hat{h}_i^2.$$

Consider now c in the interior of d_{n+1}, then

$$\begin{aligned}
-t_{n+1}(\hat{h},c) &= \sum_{i=1}^{n+1}\hat{h}_i^2-\sum_{i=1}^{n+1}\hat{h}_i^2 D_i\,g_{n+1,i}-2\sum_{i=1}^{n}\sum_{j=i+1}^{n+1}\hat{h}_i\hat{h}_j D_i\,g_{n+1,j}\\
&= \sum_{i=1}^{n}\hat{h}_i^2-[2/(a_n+1)]\sum_{i=1}^{n}\hat{h}_i^2 D_i\,g_{n,i}\\
&\quad -A_n\sum_{i=1}^{n}\hat{h}_i^2 D(i,n+1,i)+\hat{h}_{n+1}^2[1-D_{n+1}\,g_{n+1,n+1}]\\
&\quad -[2/(a_n+1)]\,2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\hat{h}_i\hat{h}_j D_i\,g_{n,j}\\
&\quad -2A_n\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\hat{h}_i\hat{h}_j D(i,n+1,j)\\
&\quad -2\hat{h}_{n+1}\sum_{i=1}^{n}\hat{h}_i D_i\,g_{n+1,n+1}\\
&= [2/(a_n+1)]\left[\sum_{i=1}^{n}\hat{h}_i^2-\sum_{i=1}^{n}\hat{h}_i^2 D_i\,g_{n,i}\right.\\
&\quad \left.-2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\hat{h}_i\hat{h}_j D_i\,g_{n,j}\right]\\
&\quad +[1-2/(a_n+1)]\sum_{i=1}^{n}\hat{h}_i^2+\hat{h}_{n+1}^2[1-A_nL_n/(a_n+1)]\\
&\quad -A_n\sum_{i=1}^{n}\hat{h}_i^2 D(i,n+1,i)-2A_n\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\hat{h}_i\hat{h}_j D(i,n+1,j)\\
&\quad -2\hat{h}_{n+1}\sum_{i=1}^{n}A_n\hat{h}_i D(i,n+1,n+1).
\end{aligned}$$

Therefore

$$-t_{n+1}(\hat{h}, c) \geqslant [2/(a_n+1)]\, 3^{1-n} \sum_{i=1}^{n} \hat{h}_i^2 + [A_n L_n/(a_n+1)] \sum_{i=1}^{n} \hat{h}_i^2$$

$$+ \hat{h}_{n+1}^2 [2/(a_n+1)] - A_n \sum_{i=1}^{n} \hat{h}_i^2 D(i, n+1, i)$$

$$-2A_n \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \hat{h}_i \hat{h}_j D(i, n+1, j)$$

$$\geqslant 2(3^{-n}) \sum_{i=1}^{n+1} \hat{h}_i^2 + \sum_{i=1}^{n} \hat{h}_i^2 [A_n [L_n/(a_n+1) - D(i, n+1, i)]]$$

$$-2A_n \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \hat{h}_i \hat{h}_j D(i, n+1, j)$$

$$\geqslant (3^{-n}) \sum_{i=1}^{n+1} \hat{h}_i^2 + \sum_{i=1}^{n} \hat{h}_i^2 \{\tfrac{1}{2}(\tfrac{1}{3})^n$$

$$+ A_n [L_n/(a_n+1) - D(i, n+1, i)]\}$$

$$+ \tfrac{1}{2}(\tfrac{1}{3})^n \sum_{i=1}^{n+1} \hat{h}_i^2 - 2A_n \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \hat{h}_i \hat{h}_j D(i, n+1, j).$$

Let us now require that A_n be chosen such that

$$\tfrac{1}{2}(\tfrac{1}{3})^n + A_n [L_n/(a_n+1) - D(i, n+1, i)] > 0$$

for $1 \leqslant i \leqslant n$ on d_{n+1}, then

$$-t_{n+1}(\hat{h}, c) \geqslant (\tfrac{1}{3})^n \sum_{i=1}^{n+1} \hat{h}_i^2 + [\tfrac{1}{2}(\tfrac{1}{3})^n] \left\{ \sum_{i=1}^{n+1} \hat{h}_i^2 \right.$$

$$\left. -2A_n \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \hat{h}_i \hat{h}_j [2(3^n)\, D(i, n+1, j)] \right\}$$

$$\geqslant (\tfrac{1}{3})^n \sum_{i=1}^{n+1} \hat{h}_i^2 + [\tfrac{1}{2}(\tfrac{1}{3})^n] \left\{ \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} [(1/n)\, \hat{h}_i^2 \right.$$

$$\left. -2\hat{h}_i \hat{h}_j A_n [2(3^n)\, D(i, n+1, j)]] \right\}$$

$$+ \tfrac{1}{2}(\tfrac{1}{3})^n \sum_{i=2}^{n+1} (i-1)\, \hat{h}_i^2/n$$

$$\geqslant (\tfrac{1}{3})^n \sum_{i=1}^{n+1} \hat{h}_i^2 + [\tfrac{1}{2}(\tfrac{1}{3})^n]$$

$$\times \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} (1/n)[\hat{h}_i - 2nA_n 3^n D(i, n+1, j)\, \hat{h}_j]^2$$

$$+ [\tfrac{1}{2}(\tfrac{1}{3})^n] \sum_{i=2}^{n+1} \hat{h}_i^2[(i-1)/n - A_n^2 \sum_{j=1}^{i-1} 4n3^{2n} D^2(j, n+1, i)].$$

Let us now require that for $2 \leqslant i \leqslant n+1$,

$$(i-1)/n - A_n^2 \sum_{j=1}^{i-1} 4n3^{2n} D^2(j, n+1, i) \geqslant 0 \qquad \text{on} \quad d_{n+1},$$

from which it follows that

$$-t_{n+1}(\hat{h}, c) \geqslant 3^{-n} \sum_{i=1}^{n+1} \hat{h}_i^2.$$

LEMMA J. *There is a positive number sequence $\{A_i\}_{i=1}^{\infty}$ such that if n is an integer greater than 2 and c is in the interior of d_n, then*

$$t_n(\hat{h}, c) + \sum_{i=1}^{n} \hat{h}_i^2 \geqslant \tfrac{1}{2}(\tfrac{1}{3})^{n-1} \sum_{i-1}^{n} A_{i-1} L_{i-1} \hat{h}_i^2.$$

Proof. For each integer $n \geqslant 2$, let

$$z_n(\hat{h}, c) = t_n(\hat{h}, c) + \sum_{i=1}^{n} \hat{h}_i^2$$

$$= \sum_{i=1}^{n} \hat{h}_i^2 D_i g_{n,i} + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \hat{h}_i \hat{h}_j D_i g_{n,j}.$$

Suppose c is in the interior of d_2, then

$$z_2(\hat{h}, c) = \hat{h}_1^2 D_1 g_{2,1} + \hat{h}_2^2 D_2 g_{2,2} + 2\hat{h}_1 \hat{h}_2 D_1 g_{2,2}$$

$$= \hat{h}_1^2[2/(a_1+1)]\, D_1 g_{1,1} + \hat{h}_2^2[A_1 L_1/(a_1+1)]$$

$$+ \hat{h}_1^2[A_1 D(1, 2, 1)] + 2\hat{h}_1 \hat{h}_2[A_1 D(1, 2, 2)]$$

and hence

$$(a_1+1)\, z_2(\hat{h}, c) = \tfrac{2}{3}\hat{h}_1^2 + A_1 L_1 \hat{h}_2^2 + A_1[(a_1+1)\, D(1, 2, 1)\, \hat{h}_1^2$$

$$+ 2\hat{h}_1 \hat{h}_2 (a_1+1)\, D(1, 2, 2)]$$

$$= \tfrac{1}{2}[\hat{h}_1^2 + A_1 L_1 \hat{h}_2^2] + \tfrac{1}{6}\hat{h}_1^2[1/2 + 6A_1(a_1+1)\, D(1, 2, 1)]$$

$$+ \tfrac{1}{12}[\hat{h}_1^2 + 6A_1 L_1 \hat{h}_2^2 + 2A_1 \hat{h}_1 \hat{h}_2 12(a_1+1)\, D(1, 2, 2)].$$

Let us require that A_1 be chosen such that

$$\tfrac{1}{2}+6A_1(a_1+1)\,D(1,2,1)>0 \qquad \text{on} \quad d_2;$$

then

$$\begin{aligned}(a_1+1)\,z_2(\hat{h},c)\geqslant \tfrac{1}{2}[\hat{h}_1^2+A_1L_1\hat{h}_2^2] &\\ +\tfrac{1}{12}[\hat{h}_1+12A_1(a_1+1)\,D(1,2,2)\,\hat{h}_2]^2 &\\ +\tfrac{1}{12}[6L_1-A_1[12(a_1+1)\,D(1,2,2)]^2]\,A_1\hat{h}_2^2. &\end{aligned}$$

Recall that $D(1,2,2)=L_1^{1/2}B(1,2,2)$, and hence

$$6L_1-A_1[12(a_1+1)\,B(1,2,2)]^2=L_1[6-A_1[12(a_1+1)\,B(1,2,2)]^2],$$

from which it follows that if A_1 is chosen such that

$$6-A_1[12(a_1+1)\,B(1,2,2)]^2>0$$

then

$$(a_1+1)\,z_2(\hat{h},c)\geqslant \tfrac{1}{2}[\hat{h}_1^2+A_1L_1\hat{h}_2^2]$$

and thus

$$z_2(\hat{h},c)\geqslant \tfrac{1}{2}(\tfrac{1}{3})[\hat{h}_1^2+A_1L_1\hat{h}_2^2] \qquad \text{on the interior of } d_2.$$

Suppose now that $A_1, ..., A_n$ have been chosen such that if $2\leqslant m\leqslant n$, then

$$z_m(\hat{h},c)\geqslant \tfrac{1}{2}(\tfrac{1}{3})^{m-1}\sum_{i=1}^{m} A_{i-1}L_{i-1}\hat{h}_i^2$$

for c in the interior of d_m.

Consider now c in the interior of d_{n+1}; then

$$\begin{aligned}z_{n+1}(\hat{h},c)=\hat{h}_{n+1}^2D_{n+1}\,g_{n+1,n+1}+2\hat{h}_{n+1}\sum_{i=1}^{n}\hat{h}_iD_i\,g_{n+1,n+1}&\\ +[2/(a_n+1)]\sum_{i=1}^{n}\hat{h}_i^2D_i\,g_{n,i}+A_n\sum_{i=1}^{n}\hat{h}_i^2D(i,n+1,i)&\\ +2[2/(a_n+1)]\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\hat{h}_i\hat{h}_jD_i\,g_{n,j}&\\ +2A_n\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\hat{h}_i\hat{h}_jD(i,n+1,j)&\end{aligned}$$

$$
\begin{aligned}
&= [A_n L_n/(a_n+1)]\,\hat{h}_{n+1}^2 + 2A_n\hat{h}_{n+1}\sum_{i=1}^{n}\hat{h}_i D(i, n+1, n+1) \\
&\quad + [2/(a_n+1)]\left[\sum_{i=1}^{n}\hat{h}_i^2 D_i g_{n,i} + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\hat{h}_i\hat{h}_j D_i g_{n,j}\right] \\
&\quad + A_n\sum_{i=1}^{n}\hat{h}_i^2 D(i, n+1, i) + 2A_n\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\hat{h}_i\hat{h}_j D(i, n+1, j) \\
&= [2/(a_n+1)]\, z_n(\hat{h}, c) \\
&\quad + [1/(a_n+1)]\left\{A_n L_n\hat{h}_{n+1}^2 + A_n\sum_{i=1}^{n}\hat{h}_i^2(a_n+1)\, D(i, n+1, i)\right. \\
&\quad \left. + 2A_n\sum_{i=1}^{n}\sum_{j=i+1}^{n+1}\hat{h}_i\hat{h}_j(a_n+1)\, D(i, n+1, j)\right\}
\end{aligned}
$$

and hence

$$
\begin{aligned}
(a_n+1)\, z_{n+1}(\hat{h}, c) &\geqslant (\tfrac{1}{3})^{n-1}\sum_{i=1}^{n} A_{i-1}L_{i-1}\hat{h}_i^2 + A_n L_n\hat{h}_{n+1}^2 \\
&\quad + A_n\sum_{i=1}^{n}\hat{h}_i^2(a_n+1)\, D(i, n+1, i) \\
&\quad + 2A_n\sum_{i=1}^{n}\sum_{j=i+1}^{n+1}\hat{h}_i\hat{h}_j(a_n+1)\, D(i, n+1, j) \\
&\geqslant \tfrac{1}{2}(\tfrac{1}{3})^{n-1}\sum_{i=1}^{n+1} A_{i-1}L_{i-1}\hat{h}_i^2 \\
&\quad + \tfrac{1}{2}(\tfrac{1}{3})^{n-1}\sum_{i=1}^{n+1} A_{i-1}L_{i-1}\hat{h}_i^2 \\
&\quad + A_n\sum_{i=1}^{n}\hat{h}_i^2(a_n+1)\, D(i, n+1, i) \\
&\quad + 2A_n\sum_{i=1}^{n}\sum_{j=i+1}^{n+1}\hat{h}_i\hat{h}_j(a_n+1)\, D(i, n+1, j) \\
&\geqslant \tfrac{1}{2}(\tfrac{1}{3})^{n-1}\sum_{i=1}^{n+1} A_{i-1}L_{i-1}\hat{h}_i^2 \\
&\quad + \sum_{i=1}^{n}\hat{h}_i^2[\tfrac{1}{4}(\tfrac{1}{3})^{n-1} A_{i-1}L_{i-1} \\
&\quad + A_n(a_n+1)\, D(i, n+1, i)]
\end{aligned}
$$

$$= \tfrac{1}{4}(\tfrac{1}{3})^{n-1} \sum_{i=1}^{n+1} A_{i-1} L_{i-1} \hat{h}_i^2$$

$$+ 2A_n \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \hat{h}_i \hat{h}_j (a_n + 1)\, D(i, n+1, j).$$

Let us require that A_n be chosen such that for $1 \leqslant i \leqslant n$,

$$\tfrac{1}{4}(\tfrac{1}{3})^{n-1} A_{i-1} + A_n(a_n + 1)\, B(i, n+1, i) > 0 \qquad \text{on} \quad d_{n+1},$$

where $D(i, n+1, i) = L_i B(i, n+1, i)$, then

$$(a_n + 1)\, z_{n+1}(\hat{h}, c) \geqslant \tfrac{1}{2}(\tfrac{1}{3})^{n-1} \sum_{i=1}^{n+1} A_{i-1} L_{i-1} \hat{h}_i^2$$

$$+ \tfrac{1}{4}(\tfrac{1}{3})^{n-1} \Bigg\{ \sum_{i=1}^{n+1} A_{i-1} L_{i-1} \hat{h}_i^2$$

$$+ 2A_n \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \hat{h}_i \hat{h}_j (\tfrac{1}{4})(\tfrac{1}{3})^{n-1} (a_n + 1)\, D(i, n+1, j) \Bigg\}.$$

Following in a manner very similar to that of the proof of Lemma I, A_n can be chosen such that

$$0 \leqslant \sum_{i=1}^{n+1} A_{i-1} L_{i-1} \hat{h}_i^2$$

$$+ 2A_n \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \hat{h}_i \hat{h}_j (\tfrac{1}{4})(\tfrac{1}{3})^{n-1} (a_n + 1)\, D(i, n+1, j)$$

on d_{n+1}, and hence

$$z_{n+1}(\hat{h}, c) \geqslant [1/(a_n + 1)](\tfrac{1}{2})(\tfrac{1}{3})^{n-1} \sum_{i=1}^{n+1} A_{i-1} L_{i-1} \hat{h}_i^2$$

$$\geqslant \tfrac{1}{2}(\tfrac{1}{3})^{n} \sum_{i=1}^{n+1} A_{i-1} L_{i-1} \hat{h}_i^2.$$

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